INTRODUCTORY NOTES 2. EQUIVALENCE RELATION

Please consider every proposition except Proposition 1.2 in this text as an exercise.

1. Equivalence relation

Definition 1.1. Suppose that X is a set and $R \subset X \times X$. We will write $x \sim_R y$ iff $(x, y) \in R$. $\sim_R$ (or R) is called equivalence relation if the following conditions hold:

(1) (reflexivity) $x \sim_R x$, $\forall x \in X$;
(2) (symmetry) $\forall x, y \in X (x \sim_R y) \implies (y \sim_R x)$;
(3) (transitivity) $\forall x, y, z \in X ((x \sim_R y) \wedge (y \sim_R z)) \implies (x \sim_R z)$.

The next statement can be taken for granted.

Proposition 1.2. *Suppose that X is a set and R is an equivalence relation. Then there exists a family of pairwise non-intersecting subsets $\{X_\alpha \mid \alpha \in I\}$ such that*

$$X = \bigsqcup_{\alpha \in I} X_\alpha, \tag{1.1}$$

and

$$\forall x, y \in X \quad ((x \sim_R y) \iff (\exists \alpha \in I,\ x, y \in X_\alpha)) \tag{1.2}$$

The subsets X_α are called the equivalence classes (of $\sim_R$). The set of equivalence classes is denoted by $X/\sim_R$.

Proposition 1.3. *Suppose that*

$$X = \bigsqcup_{\alpha \in I} X_\alpha. \tag{1.3}$$

Define $R \subset X \times X$ by

$$((x, y) \in R) \iff (\exists \alpha \in I,\ x, y \in X_\alpha). \tag{1.4}$$

Then R is an equivalence relation.

Definition 1.4. Suppose that X is a set and R is an equivalence relation. Define the canonical projection map

$$p : X \to X/\sim_R \tag{1.5}$$

as the map which sends each $x \in X$ to its equivalence class.

Exercise 1.5. Which of the following are equivalence relations:

(a)
$$X := \mathbb{Z}, \quad (x \sim_R y) \iff (x - y \geq 0); \tag{1.6}$$

(b)
$$X := \mathbb{Z}, \quad (x \sim_R y) \iff (x - y \text{ is even}); \tag{1.7}$$

(c)
$$X := \mathbb{Z}, \quad (x \sim_R y) \iff (x + y \text{ is even}); \tag{1.8}$$

(d)
$$X := \mathbb{R}, \quad (x \sim_R y) \iff (x - y \in \mathbb{Z}); \tag{1.9}$$

Date: January 29, 2026.

(e)

$$X := \mathbb{R}, \quad (x \sim_R y) \iff (x + y \in \mathbb{Z}); \tag{1.10}$$

(f)

$$X := \mathbb{R}^2, \quad \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \sim_R \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \iff (x_1 + y_1 = x_2 + y_2); \tag{1.11}$$

(g)

$$X := \mathbb{R}^2, \quad \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \sim_R \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \iff (x_1^2 + y_1^2 = x_2^2 + y_2^2); \tag{1.12}$$

(h)

$$X := \mathbb{R}^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.13}$$

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \sim_R \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \iff \left(\exists a \in \mathbb{R} \setminus \{0\}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = a \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right); \tag{1.14}$$

(i)

$$X := \{\text{the set of differentiable functions from } \mathbb{R} \text{ to } \mathbb{R}\}, \tag{1.15}$$

$$(f \sim_R g) \iff (f(0) = g(0), f'(0) = g'(0)); \tag{1.16}$$

(j)

$$X := \{\text{the set of differentiable functions from } \mathbb{R} \text{ to } \mathbb{R}\}, \tag{1.17}$$

$$(f \sim_R g) \iff (f(0) = g(0), f(1) = g(1)); \tag{1.18}$$

(k)

$$X := \{\text{the set of Riemann integrable functions from } [0,1] \text{ to } \mathbb{R}\}, \tag{1.19}$$

$$(f \sim_R g) \iff \left(\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx\right). \tag{1.20}$$

2. Simple Commutative Diagrams

Proposition 2.1. *Let $f : X \to Y$ be a map. Define $R \subset X \times X$ by*

$$((x, y) \in R) \iff (f(x) = f(y)). \tag{2.1}$$

This is an equivalence relation. Denote the set of its equivalence classes by X/f, and the canonical projection by p. There exists a unique map $s : X/f \to Y$ such that $s \circ p = f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p} & \nearrow{\scriptstyle s} & \\ X/f & & \end{array} \tag{2.2}$$

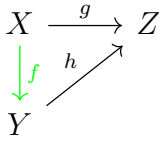
The map s is injective. It is a natural bijection of X/f and $f(X)$. This construction shows that every map can be represented as a composition of an injective map and a surjective map.

Exercise 2.2. Suppose that $g : X \to Z$ and $h : X \to Y$ are maps. Formulate a necessary and sufficient condition for a map $f : Y \to Z$ to exist.

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle h} & \nearrow{\scriptstyle f} & \\ Y & & \end{array} \tag{2.3}$$

Formulate a condition equivalent to the uniqueness of f.

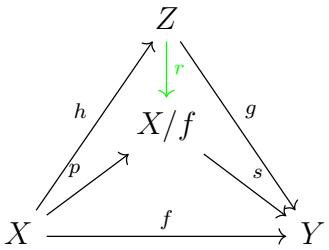
Exercise 2.3. Suppose that $g : X \to Z$ and $h : Y \to Z$ are maps. Formulate a necessary and sufficient condition for a map $f : X \to Y$ to exist.



(2.4)

Formulate a condition equivalent to the uniqueness of f.

Proposition 2.4. *We use the notation from Proposition 2.1. Suppose that Z is a set, $h : X \to Z$ and $g : Z \to Y$ are maps such that h is surjective and $g \circ h = f$. Then there exists a unique map $r : Z \to X/f$ such that $r \circ h = p$, and $s \circ r = g$.*



(2.5)

Explain in what sense this property uniquely determines X/f.