LECTURE 4. CHANGE OF BASIS.

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please treat every proposition in this lecture as an exercise.

1. Change of Bases for Linear Maps

Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V. Recall that in the previous lecture, we introduced the notation $\varphi_\alpha(x)$ for the linear map from V to $\mathbb{F}^n$ assigning to a vector x its coordinates in a basis α. Also, we used the notation $[A]^\beta_\alpha$ for the matrix of a linear map A expressing the images of the vectors of the basis α as linear combinations of the vectors of the basis β. In our current notation, it takes the following form:

$$Av_i = \sum_{j=1}^{n} a_i^j v'_j, \quad \forall 1 \le i \le n, \quad [A]^\beta_\alpha = (a_i^j)_{1\le i,j\le n}. \tag{1.1}$$

Before defining the change-of-basis matrix, we will establish some of its properties.

Proposition 1.1. *Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V.*

a) Then the matrix $[\mathrm{Id}]^\beta_\alpha$ is invertible.

b) The matrix $[\mathrm{Id}]^\beta_\alpha$ is equal to the matrix of the linear operator $\varphi_\beta \circ (\varphi_\alpha)^{-1}$ in the standard basis of $\mathbb{F}^n$.

Definition 1.2. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V. The matrix $[\mathrm{Id}]^\beta_\alpha$ is called the change of basis matrix from α to β.

From Proposition 1.1, we see that the coordinates of a vector in one basis can be obtained from its coordinates in the other by applying the change-of-basis matrix:

$$\varphi_\beta(x) = [\mathrm{Id}]^\beta_\alpha \varphi_\alpha(x). \tag{1.2}$$

Recall the following rule connecting the composition of linear maps with matrix multiplication.

$$[B]^\gamma_\beta \cdot [A]^\beta_\alpha := [BA]^\gamma_\alpha. \tag{1.3}$$

We use it to establish the following claim.

Proposition 1.3. *Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V, $\gamma = (u_1, \ldots, u_m)$, and $\delta = (u'_1, \ldots, u'_m)$ are two ordered bases of a vector space U, C_1 is the change of basis matrix from α to β, C_2 is the change of basis matrix from γ to δ, and $A \in \mathrm{Hom}(V, U)$. Then the matrices $[A]^\gamma_\alpha$ and $[A]^\delta_\beta$ satisfy the following relation:*

$$\boxed{[A]^\delta_\beta = C_2\, [A]^\gamma_\alpha\, C_1^{-1}.} \tag{1.4}$$

Let us consider the case of linear operators. We agree that for an operator $A \in \mathrm{End}(V)$, we pick a single basis α in V. So the matrices of endomorphisms A have the form $[A]^\alpha_\alpha$ for some basis α. Then, we may reformulate Proposition 1.3 for that case.

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Proposition 1.4. *Suppose that $\alpha = (v_1, \dots, v_n)$ and $\beta = (v'_1, \dots, v'_n)$ are two ordered bases of a vector space V, C is the change of basis matrix from α to β, and $A \in \mathrm{End}(V)$. Then the matrices $[A]^\alpha_\alpha$ and $[A]^\beta_\beta$ satisfy the following relation:*

$$\boxed{[A]^\beta_\beta = C[A]^\alpha_\alpha C^{-1}.} \tag{1.5}$$

2. Equivalence of Matrices

Following the result of Proposition 1.3, we define the equivalence of matrices.

Definition 2.1. Suppose that $A, B \in \mathrm{Mat}(m \times n, \mathbb{F})$. They are called equivalent if there exist invertible matrices $C \in \mathrm{Mat}(m \times m, \mathbb{F})$, $D \in \mathrm{Mat}(n \times n, \mathbb{F})$ such that

$$A = C\,B\,D^{-1}. \tag{2.1}$$

The analogous notion for linear operators is called similarity of square matrices.

Definition 2.2. Suppose that $A, B \in \mathrm{Mat}(n \times n, \mathbb{F})$. They are called similar if there exists an invertible matrix $C \in \mathrm{Mat}(n \times n, \mathbb{F})$ such that

$$A = C\,B\,C^{-1}. \tag{2.2}$$

Using the ideas behind the First Isomorphism Theorem, we can essentially solve the problem of classification of equivalent matrices.

Proposition 2.3. *Suppose that V and U are finite-dimensional vector spaces, $A \in \mathrm{Hom}(V, U)$, and $\mathrm{rk}(A) = k$. Then there exist bases α and β of the spaces V and U, respectively, such that the matrix of the linear map has the following block diagonal form:*

$$\boxed{[A]^\beta_\alpha = \begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & 0 \end{pmatrix},} \tag{2.3}$$

where Id_k is the $k \times k$ identity matrix.

Therefore, we see that every equivalence class of matrices of the same size consists of the matrices of the same rank.

The problem of classification of similar matrices turns out to be more complicated. We are going to deal with it later in the course.

3. System of Linear Equations

A system of linear equations

$$\sum_{i=1}^{n} a^j_i x_i = b_j, \quad 1 \le j \le m \tag{3.1}$$

can be represented as $Ax = b$, where

$$A = (a^j_i)_{1 \le i \le n,\, 1 \le j \le m}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}. \tag{3.2}$$

This notation allows us to formulate two general statements about those systems. We are going to view A both as a matrix and as a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ defined by that matrix in the standard bases.

Proposition 3.1. *Solutions of the system $Ax = 0$ form a subspace in the vector space $\mathbb{F}^n$. The dimension of the subspace of solutions of the system $Ax = 0$ equals $n - \mathrm{rk}(A)$. In particular, if $m < n$, then there exists a non-zero solution to the homogeneous system.*

Proposition 3.2. *A solution of the system $Ax = b$ exists if and only if $b \in \mathrm{im}A$. In particular, if $\mathrm{rk}(A) = m$, then a solution exists. The difference between any two solutions of the system $Ax = b$ is itself a solution of the system $Ax = 0$.*

4. Dual Space

Definition 4.1. Let V be a vector space over a field $\mathbb{F}$. The vector space $\mathrm{Hom}(V, \mathbb{F})$ is called the dual space (to the vector space V) and is denoted by V^*. The elements of V^* are called linear functionals, or covectors, or linear forms.

Definition 4.2. Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V. Define $f^i \in V^*$ at the basis vectors via

$$\boxed{f^i(v_j) = \delta^i_j, \quad 1 \le i, j \le n,} \tag{4.1}$$

where δ^i_j is the Kronecker delta:

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

The set $\{f^1, f^2, \ldots, f^n\}$ is called the dual basis (to the basis $\{v_1, \ldots, v_n\}$).

Exercise 4.3. Show that $\{f^1, f^2, \ldots, f^n\}$ is a basis of V^*.

Proposition 4.4. *Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Then there exists an isomorphism of V and $(V^*)^*$ such that it does not depend on the choice of basis in V.*

Exercise 4.5. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V and C is the change of basis matrix from α to β. Denote by γ and δ the bases of V^* dual to α and β, respectively. Express the change-of-basis matrix from γ to δ in terms of the matrix C.

Definition 4.6. Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{Hom}(V, U)$. Define $A^* \in \mathrm{Hom}(U^*, V^*)$ by

$$\boxed{(A^*(f))(v) = f(Av), \quad \forall v \in V, \forall f \in U^*.} \tag{4.3}$$

A^* is called the operator adjoint of A.

Exercise 4.7. Show that for any $A \in \mathrm{Hom}(V, U)$ the map $A^* \in \mathrm{Hom}(U^*, V^*)$ exists, and it is unique.

Exercise 4.8. Suppose that V and U are finite-dimensional vector spaces, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, $\gamma = \{f^1, \ldots, f^n\}$ is its dual basis, $\beta = \{u_1, \ldots, u_m\}$ is a basis of U, $\delta = \{g^1, \ldots, g^m\}$ is its dual basis, $A \in \mathrm{Hom}(V, U)$ and its matrix $[A]^\beta_\alpha$ equals

$$(a^j_i)_{1 \le i \le n, 1 \le j \le m} \tag{4.4}$$

Find the matrix of $[A^*]^\gamma_\delta$.

5. Symmetric Group

Definition 5.1. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if ρ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called the symmetric group and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{5.1}$$

A more common notation is a cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{5.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{5.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, m elements, we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{5.4}$$

One can show that the cycle notation is well-defined. A cycle of length 2 is called a transposition.

Here is an example of the product of two permutations written in cycle notation:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{5.5}$$

Exercise 5.2. Show that S_n is a group with respect to composition and find its order.

One may compare the following statement with the change of basis for linear operators.

Exercise 5.3. Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{5.6}$$

, where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Exercise 5.4. Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{5.7}$$

Show that ψ_ρ is an automorphism of the group S_n.

Exercise 5.5. Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.

Proposition 5.6. *For every positive integer n, there exists a unique map* sgn $: S_n \to \{1, -1\}$ *such that*

(1) $\operatorname{sgn}((i, j)) = -1, \quad \forall 1 \le i < j \le n;$

(2) $\mathrm{sgn}(\rho\sigma) = \mathrm{sgn}(\rho)\mathrm{sgn}(\sigma)$.

The value $\mathrm{sgn}(\rho)$ *is called the sign of the permutation* ρ.

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{5.8}$$

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 2.5 - 2.6;
- Roman, pp.65 - 72, 94 - 101;
- Axler, Sections 3D, 3F.