LECTURE 5. TENSOR PRODUCTS

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please consider the proof of every proposition in this lecture to be an exercise.

1. Dual Space

Definition 1.1. Let V be a vector space over a field $\mathbb{F}$. The vector space $\mathrm{Hom}(V, \mathbb{F})$ is called the dual space (to the vector space V) and is denoted by V^*. The elements of V^* are called linear functionals, or covectors, or linear forms.

Definition 1.2. Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V. Define $f^i \in V^*$ on the basis vectors via

$$\boxed{f^i(v_j) = \delta^i_j, \quad 1 \leq i, j \leq n,} \tag{1.1}$$

where δ^i_j is the Kronecker delta:

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The set $\{f^1, f^2, \ldots, f^n\}$ is called the dual basis (to the basis $\{v_1, \ldots, v_n\}$).

Exercise 1.3. Show that $\{f^1, f^2, \ldots, f^n\}$ is a basis of V^*.

Proposition 1.4. *Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Then there exists an isomorphism of V and $(V^*)^*$ such that it does not depend on the choice of basis in V.*

Exercise 1.5. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v'_1, \ldots, v'_n)$ are two ordered bases of a vector space V and C is the change of basis matrix from α to β. Denote by γ and δ the bases of V^* dual to α and β, respectively. Express the change-of-basis matrix from γ to δ in terms of the matrix C.

Definition 1.6. Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{Hom}(V, U)$. Define $A^* \in \mathrm{Hom}(U^*, V^*)$ by

$$\boxed{(A^*(f))(v) = f(Av), \quad \forall v \in V, \forall f \in U^*.} \tag{1.3}$$

A^* is called the operator adjoint of A.

Exercise 1.7. Show that for any $A \in \mathrm{Hom}(V, U)$ the map $A^* \in \mathrm{Hom}(U^*, V^*)$ exists, and it is unique.

Exercise 1.8. Suppose that V and U are finite-dimensional vector spaces, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, $\gamma = \{f^1, \ldots, f^n\}$ is its dual basis, $\beta = \{u_1, \ldots, u_m\}$ is a basis of U, $\delta = \{g^1, \ldots, g^m\}$ is its dual basis, $A \in \mathrm{Hom}(V, U)$ and its matrix $[A]^\beta_\alpha$ equals

$$(a^j_i)_{1 \leq i \leq n, 1 \leq j \leq m} \tag{1.4}$$

Find the matrix $[A^*]^\gamma_\delta$.

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2. Tensor Products of Vector Spaces

Definition 2.1. Assume that V and U are vector spaces over a field $\mathbb{F}$. Denote by $Free(V,U)$ the set of all formal finite linear combinations of elements of $V \times U$:

$$Free(V,U) := \left\{ \sum_{i=1}^{n} a_i(x_i, y_i) \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\}. \tag{2.1}$$

(Note that the pairs (x_i, y_i) are simply symbols here. Probably it would be less confusing if we wrote $Free(V,U)$ as

$$\left\{ \sum_{i=1}^{n} a_i e_{(x_i,y_i)} \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\} \tag{2.2}$$

but then we would get more cumbersome expressions further in this definition.)

The set $Free(V,U)$ has the natural structure of a vector space. Now we may define a subspace $Ident(V,U) \subset Free(V,U)$ as

$$\begin{aligned} Ident(V,U) = \operatorname{span}\langle \{(ax+bz,y) - a(x,y) - b(z,y) \,|\, x,z \in V, y \in U, a,b \in \mathbb{F}\} \cup \\ \{(x, ay+bt) - a(x,y) - b(x,t) \,|\, x \in V, y,t \in U, a,b \in \mathbb{F}\} \rangle. \end{aligned} \tag{2.3}$$

The tensor product of vector spaces V and U is, by definition, the quotient space $Free(V,U)$ by $Ident(V,U)$. We denote the tensor product of vector spaces V and U by $V \otimes U$. Also we denote by

$$\sum_{i=1}^{n} a_i x_i \otimes y_i \tag{2.4}$$

the equivalence class that contains the element $\sum_{i=1}^{n} a_i(x_i, y_i)$. A map $\operatorname{emb} : V \times U \to V \otimes U$ is defined by $\operatorname{emb}((v,u)) := v \otimes u$. The elements of $V \otimes U$ are often called tensors.

For example, the element $(a \cdot x + 0 \cdot z, y) - a(x,y) - 0 \cdot (z,y) \in Ident(V,U)$. Therefore, $(ax) \otimes y = a(x \otimes y)$, $\forall a \in \mathbb{F}$. In particular, $0 \otimes y = 0$.

The following statement may be considered as a property characterizing the tensor product of two vector spaces.

Proposition 2.2. *Suppose that V, U and W are vector spaces over a field $\mathbb{F}$, and a map $f : V \times U \to W$ satisfies the following conditions:*

$$\begin{aligned} f(ax+bz,y) &= af(x,y) + bf(z,y), \\ f(x,ay+bt) &= af(x,y) + bf(x,t), \quad \forall x,z \in V, \forall y,t \in U, \forall a,b \in \mathbb{F}. \end{aligned} \tag{2.5}$$

Then there exists a unique map $A \in \operatorname{Hom}(V \otimes U, W)$ such that $A \circ \operatorname{emb} = f$.

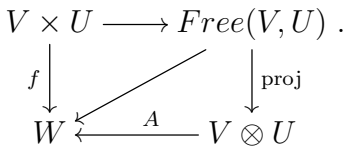
$$\begin{array}{ccc} V \times U & \xrightarrow{\text{emb}} & V \otimes U \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle A} & \\ W & & \end{array}$$

To prove Proposition 2.2, we will establish the following lemma first.

Lemma 2.3. *Suppose that V, W are vector spaces, U is a subspace of V, by $\operatorname{proj} : V \to V/U$ we denote the canonical projection, and $B \in \operatorname{Hom}(V,W)$. Then, the existence of a map $C \in \operatorname{Hom}(V/U, W)$ such that $C \circ \operatorname{proj} = B$ is equivalent to the condition $U \subseteq \ker(B)$.*

$$\begin{array}{ccc} V & \xrightarrow{B} & W \\ \downarrow{\scriptstyle \text{proj}} & \nearrow{\scriptstyle C} & \\ V/U & & \end{array}$$

Returning to the proof of Proposition 2.2, we note that the diagram defines the map $g : Free(V,U) \to W$. Then, we may apply Lemma 2.3 to the lower left triangle of the diagram



Proposition 2.4. *Assume that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \dots, v_n\}$ is a basis of V, $\{u_1, \dots, u_m\}$ is a basis of U. Then*

$$\{v_i \otimes u_j \,|\, 1 \le i \le n, 1 \le j \le m\} \tag{2.6}$$

is a basis of $V \otimes U$.

One may use the result of Proposition 2.2 in the proof of linear independence of the set in (2.6).

Proposition 2.5. *Assume that V and U are finite-dimensional vector spaces. Define $\varphi : V^* \otimes U \to \mathrm{Hom}(V,U)$ by*

$$\varphi(f \otimes u)(v) := f(v)u, \quad \forall v \in V, u \in U, f \in V^*. \tag{2.7}$$

The map φ is well-defined and $\varphi \in Iso(V^* \otimes U, \mathrm{Hom}(V,U))$.

Note that in the Proposition 2.5, we have defined the map only on the tensors of the form $v \otimes u$. That is not uncommon. We are following Proposition 2.2 here.

As a special case of Proposition 2.5, we may obtain an important corollary.

Proposition 2.6. *Assume that V is a finite-dimensional vector space. Then the vector spaces $V \otimes V^*$ and $\mathrm{End}(V)$ are canonically isomorphic.*

Definition 2.7. Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(U)$. Define the tensor product of linear operators $A \otimes B \in \mathrm{End}(V \otimes U)$ by the formula

$$(A \otimes B)(v \otimes u) := (Av) \otimes (Bu) \tag{2.8}$$

on the tensors of the form $v \otimes u$ and then extend by linearity. By definition, we put

$$A^{\otimes k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}} \tag{2.9}$$

Exercise 2.8. Show that $A \otimes B$ is well-defined.

Exercise 2.9. Suppose that $A, B \in \mathrm{End}(V)$. Is it always true that

$$(AB)^{\otimes k} = ((A)^{\otimes k})((B)^{\otimes k})? \tag{2.10}$$

Exercise 2.10. Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \dots, v_n\}$ is a basis of V, $\{u_1, \dots, u_m\}$ is a basis of U, $A \in \mathrm{End}(V)$, its matrix in the basis $\{v_1, \dots, v_n\}$ is

$$(a_i^j)_{1 \le i \le n, 1 \le j \le n}, \tag{2.11}$$

$B \in \mathrm{End}(U)$, its matrix in the basis $\{u_1, \dots, u_m\}$ is

$$(b_i^j)_{1 \le i \le m, 1 \le j \le m}. \tag{2.12}$$

Find the matrix of $A \otimes B$ in the basis

$$\{v_1 \otimes u_1, v_1 \otimes u_2, \dots, v_1 \otimes u_m, v_2 \otimes u_1, \dots, v_n \otimes u_m\}. \tag{2.13}$$

3. Symmetric Group

Definition 3.1. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if ρ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called the symmetric group and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{3.1}$$

A more common notation is cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{3.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{3.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, m elements we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{3.4}$$

One may show that cycle notation is well-defined. A cycle of length 2 is called a transposition.

Here is an example of the product of two permutations written in cycle notation:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{3.5}$$

Exercise 3.2. Show that S_n is a group with respect to composition and find its order.

One may compare the following statement with the change of basis for linear operators.

Exercise 3.3. Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{3.6}$$

where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Exercise 3.4. Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{3.7}$$

Show that ψ_ρ is an automorphism of the group S_n.

Exercise 3.5. Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.

Proposition 3.6. *For every positive integer n there exists a unique map* sgn $: S_n \to \{1, -1\}$ *such that*

(1) $\operatorname{sgn}((i,j)) = -1, \quad \forall 1 \le i < j \le n;$
(2) $\operatorname{sgn}(\rho\sigma) = \operatorname{sgn}(\rho)\operatorname{sgn}(\sigma).$

The value $\operatorname{sgn}(\rho)$ *is called the sign of the permutation* ρ.

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{3.8}$$

Chapters from the textbooks relevant to the lecture:

- Roman, pp. 363-378;
- Axler, Sections 3F, 9D.