LECTURE 6. SYMMETRIC AND EXTERIOR POWERS. POLYNOMIALS

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please consider the proof of every proposition in this lecture to be an exercise.

1. Symmetric and Exterior Powers

Definition 1.1. Let V be a vector space over $\mathbb{F}$.

Define

$$V^{\otimes k} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}}. \tag{1.1}$$

Define the k-th symmetric and exterior powers of V, $S^k(V)$ and $\Lambda^k(V)$, respectively, as the following subspaces of $V^{\otimes k}$:

$$S^k(V) := \operatorname{span}\langle \sum_{\rho \in S_k} x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V \rangle, \tag{1.2}$$

$$\Lambda^k(V) := \operatorname{span}\langle \sum_{\rho \in S_k} \operatorname{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V \rangle. \tag{1.3}$$

We use the notation

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k := \frac{1}{k!} \sum_{\rho \in S_k} \operatorname{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)}. \tag{1.4}$$

Exercise 1.2. Suppose that $\dim V = n$. Find $\dim S^k(V)$ and $\dim \Lambda^k(V)$.

Proposition 1.3. *Let $A \in \operatorname{End}(V)$, then*

$$\forall z \in S^k(V),\ A^{\otimes k}(z) \in S^k(V), \tag{1.5}$$

$$\forall z \in \Lambda^k(V),\ A^{\otimes k}(z) \in \Lambda^k(V). \tag{1.6}$$

Definition 1.4. Let $A \in \operatorname{End}(V)$. Denote the restrictions of $A^{\otimes k}$ to the subspaces $S^k(V)$ and $\Lambda^k(V)$ by $S^k(A) \in \operatorname{End}(S^k(V))$ and by $\Lambda^k(A) \in \operatorname{End}(\Lambda^k(V))$, respectively.

2. Determinant

Definition 2.1. Assume that V is a finite-dimensional vector space, $A \in \operatorname{End}(V)$. Note that $\dim(\Lambda^{\dim V}(V)) = 1$. Therefore, $\Lambda^{\dim V}(A)$ is a multiple of the identity map. We will define the determinant $\det(A)$ of the linear operator A by the following equality

$$\boxed{\Lambda^{\dim V}(A) = \det(A) \cdot \mathrm{Id}.} \tag{2.1}$$

Proposition 2.2. *Suppose that $\dim V < \infty$ and $A, B \in \operatorname{End}(V)$. Then*

$$\boxed{\det(AB) = \det(A) \cdot \det(B).} \tag{2.2}$$

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Proposition 2.3. *Suppose that* $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$ *has a matrix* $(a_i^j)_{1\le i,j\le n}$ *in some basis. Then*

$$\boxed{\det(A) = \sum_{\rho\in S_n} \mathrm{sgn}(\rho) \prod_{j=1}^{n} a^j_{\rho(j)}.} \tag{2.3}$$

3. Decomposition of polynomials

Definition 3.1. A polynomial in a variable x is a formal sum of the form

$$p = \sum_{j=0}^{k} a_j x^j, \quad k \ge 0,\ a_j \in \mathbb{F}. \tag{3.1}$$

We assume that two sums of that form are equal if they differ only by some monomials with zero coefficients. If $a_k \neq 0$ we say that the degree of the polynomial p, denoted $\deg(p)$, is k. In that case, we are going to call a_k the leading coefficient of the polynomial p. If all coefficients of the polynomial equal zero then we put $\deg(0) := -\infty$. The set of polynomials is a vector space with respect to standard addition and scalar multiplication. We denote that vector space by $\mathbb{F}[x]$. Also, we put

$$\mathbb{F}_n[x] := \{p \in \mathbb{F}[x] \mid \deg(p) \le n\}. \tag{3.2}$$

With every polynomial $p \in \mathbb{F}[x]$ we associate a function $p : \mathbb{F} \to \mathbb{F}$.

Note that $\forall a \in \mathbb{F}, a \neq 0$ we have that $\deg(a) = 0$. Next we may define Euclidean division for polynomials.

Proposition 3.2. *For every two polynomials* $f, g \in \mathbb{F}[x], g \neq 0$ *there exist unique polynomials* $q, r \in \mathbb{F}[x]$ *such that the following conditions hold:*

(1) $f = q \cdot g + r$;
(2) $\deg(r) < \deg g$.

Definition 3.3. We will say that a polynomial $g \in \mathbb{F}[x], g \neq 0$ divides a polynomial $f \in \mathbb{F}[x]$ if there exists a polynomial $q \in \mathbb{F}[x]$ such that $f = g \cdot q$. We are going to write $g \mid f$. Note that $\deg(f) \ge \deg(g)$ in that case.

For arbitrary polynomials $f_1, \dots, f_m \in \mathbb{F}[x]$ we are going to denote their greatest common divisor by $GCD(f_1, \dots, f_m)$. The GCD is determined up to multiplication by a non-zero scalar, so we will assume that its leading coefficient equals one to ensure uniqueness.

Example 3.4. Suppose that

$$p_1(x) = x^3 - x, \quad p_2(x) = x^3 - x^2 + x - 1, \quad p_3(x) = x^2 - 1. \tag{3.3}$$

Then

$$p_3 \mid p_1, \quad p_3 \nmid p_2, \quad GCD(p_1, p_2, p_3) = x - 1. \tag{3.4}$$

Exercise 3.5. Show that

$$((x - a) \mid f) \quad \Longleftrightarrow \quad (f(a) = 0). \tag{3.5}$$

Now we are going to state an important property that has its analogs for integers and, more generally, for principal ideals domains.

Proposition 3.6 (Bézout's identity for polynomials)**.** *Suppose that* $f_1, \dots, f_m \in \mathbb{F}[x]$ *and* $g = GCD(f_1, \dots, f_m)$*. Then there exist* $p_1, \dots, p_m \in \mathbb{F}[x]$ *such that*

$$g = \sum_{j=1}^{m} p_j \cdot f_j. \tag{3.6}$$

To prove that result, we may consider the set

$$I := \{\sum_{j=1}^{m} p_j \cdot f_j \mid p_j \in \mathbb{F}[x]\} \tag{3.7}$$

and put g to be a non-zero element of I of minimal degree. Again it is defined up to multiplication by a non-zero scalar.

Proposition 3.6 allows us to prove the following lemma.

Lemma 3.7. *Suppose that* $g, p, q \in \mathbb{F}[x]$*,* $g \mid pq$ *and* $GCD(g, p) = 1$*. Then* $g \mid q$*.*

Definition 3.8. A polynomial f is called irreducible if

(1) $\deg(f) \geq 1$;
(2) for every $g, q \in \mathbb{F}[x]$ such that $f = g \cdot q$ we have that either $\deg(g) = 0$ or $\deg(q) = 0$.

From Lemma 3.7 we may deduce the unique factorization property for polynomials in one variable.

Proposition 3.9. *Every polynomial* $f \in \mathbb{F}[x]$ *may be factored into a product of a non-zero constant and a finite number of irreducible polynomials:*

$$f = a \prod_{j=1}^{m} q_j, \quad a \in \mathbb{F}, q_j \in \mathbb{F}[x]. \tag{3.8}$$

This decomposition is unique up to the permutation of the factors and the multiplication of the factors by non-zero constants whose product is 1. Again we may assume that the leading coefficient of every q_j *equals one to ensure uniqueness.*

Example 3.10. We have the following decompositions

$$2x^3 + x \in \mathbb{R}[x], \quad 2x^3 + x = 2x(x^2 + 1); \tag{3.9}$$

$$2x^3 + x \in \mathbb{C}[x], \quad 2x^3 + x = 2x(x + i)(x - i). \tag{3.10}$$

We are going to take for granted the next claim.

Proposition 3.11 (Fundamental Theorem of Algebra)**.** *Every irreducible complex polynomial equals* $(x - z)$ *for some* $z \in \mathbb{C}$*.*

From the complex case we may deduce the analogous result for the field $\mathbb{R}$.

Proposition 3.12. *Every irreducible real polynomial equals either* $(x - a)$ *for some* $a \in \mathbb{R}$*, or* $x^2 + bx + c$*, where* $b^2 - 4c < 0$*.*

It is useful to consider the conjugation in the field $\mathbb{C}$ to prove this proposition.

Chapters from the textbooks relevant to the lecture:

- Roman, pp. 363-378, 393-395,
- Axler, Chapter 4.