LECTURE 8. EIGENVALUES AND EIGENVECTORS. PRIMARY DECOMPOSITION THEOREM.

Please treat the proof of every proposition in this lecture as an exercise. The notation $\mathbb{F}$ stands for a field.

1. Eigenvalues and Eigenvectors.

In this section, we will consider the case when an invariant subspace is one-dimensional.

Definition 1.1. Suppose that A is a linear operator in a vector space V. An element $v \in V$ is called an eigenvector of the endomorphism A if:

(1) $v \neq 0$;
(2) $\exists \lambda \in \mathbb{F}$ such that $Av = \lambda v$.

The number λ is called an eigenvalue of the linear operator A. The set of all eigenvalues of A is called the spectrum of A and is denoted by $\mathrm{Spec}(A)$.

Exercise 1.2. Find $\mathrm{Spec}(\frac{d}{dx})$ in the vector space $\mathbb{R}_n[x]$.

Now we will describe an algorithm for finding all eigenvalues of a given linear operator.

Proposition 1.3. *Suppose that $A \in \mathrm{End}(V)$. Then*

$$(\lambda \text{ is an eigenvalue of } A) \iff (\lambda \text{ is a root of the characteristic polynomial } \chi_A) \tag{1.1}$$

Definition 1.4. Suppose that A is a linear operator in a vector space V and λ is an eigenvalue of A. Define

$$V_\lambda := \{x \in V \mid Ax = \lambda x\}. \tag{1.2}$$

V_λ is called an eigenspace of A. We see that V_λ is a subspace of the vector space V. Its dimension $\dim V_\lambda$ is called the geometric multiplicity of λ.

Note that V_λ consists of the zero vector and all eigenvectors with the eigenvalue λ.

If we know all roots of the characteristic polynomial of A, then we may find all eigenvectors of the linear operator A by solving the systems of linear equations $(A - \lambda \mathrm{Id})(x) = 0$ for every root λ. If $\dim V < \infty$ and we fix a basis of V, then for every fixed λ, $(A - \lambda \mathrm{Id})(x) = 0$ is a homogeneous system of linear equations. Hence,

$$\dim V_\lambda = \dim V - \mathrm{rk}(A - \lambda \mathrm{Id}). \tag{1.3}$$

Exercise 1.5. How can we describe all linear operators $A \in \mathrm{End}(V)$ in a finite-dimensional vector space for which the geometric multiplicity of some eigenvalue equals $\dim V$?

Definition 1.6. Suppose that A is a linear operator in a finite-dimensional vector space V. Consider the factorization of the characteristic polynomial χ_A into irreducible polynomials. The multiplicity of the factor $(x - \lambda)$ in this product is called the algebraic multiplicity of λ.

Proposition 1.7. *Suppose that A is a linear operator in a finite-dimensional vector space V, and λ is an eigenvalue of A. Then the geometric multiplicity of λ is not greater than its algebraic multiplicity.*

Exercise 1.8. Can the geometric multiplicity of some eigenvalue be equal to its algebraic multiplicity?

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Exercise 1.9. Suggest an example of a linear operator such that the geometric multiplicity of some eigenvalue is strictly less than its algebraic multiplicity.

2. Diagonalization.

Proposition 2.1. *Suppose that $v_1, \dots, v_k$ are eigenvalues of a linear operator A with distinct eigenvalues. Then the set $\{v_1, \dots, v_k\}$ is linearly independent.*

From Proposition 2.1, we may deduce a useful result about the corresponding eigenspaces.

Proposition 2.2. *Suppose that A is a linear operator in a finite-dimensional vector space V, and $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A. Then the sum of the corresponding eigenspaces*

$$V_{\lambda_1} + \dots + V_{\lambda_k} \tag{2.1}$$

is a direct sum.

Definition 2.3. A square matrix

$$B = (b_i^j)_{1 \le i,j \le n} \in \mathrm{Mat}(n \times n, \mathbb{F}) \tag{2.2}$$

is called diagonal if $b_i^j = 0$ for all $i \neq j$:

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}. \tag{2.3}$$

A linear operator A in a finite-dimensional vector space V is called diagonalizable if and only if there exists a basis of V in which the matrix of A is diagonal.

Exercise 2.4. Show that a linear operator defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalizable.

Definition 2.5. We say that a polynomial $p \in \mathbb{F}[x]$ splits if it can be represented as

$$p(x) = c \cdot \prod_{j=1}^{k} (x - a_j)^{m_j} \tag{2.4}$$

for some $a_1, \dots, a_k, c \in \mathbb{F}$, $m_1, \dots, m_k \in \mathbb{Z}_+$.

Now we will state an obvious property of diagonalizable linear operators.

Proposition 2.6. *If a linear operator A in a finite-dimensional vector space V is diagonalizable, then its characteristic polynomial splits.*

Exercise 2.7. Suppose that the characteristic polynomial of an operator A in a finite-dimensional vector space V splits. Does A have to be diagonalizable?

Using Proposition 2.1, we can obtain a sufficient condition for diagonalizability.

Proposition 2.8. *If the characteristic polynomial of a linear operator A in a finite-dimensional vector space V has $\dim V$ distinct roots, then A is diagonalizable.*

Also, we may easily formulate a statement about triangularization of a matrix.

Definition 2.9. A square matrix

$$B = (b_i^j)_{1 \le i,j \le n} \in \mathrm{Mat}(n \times n, \mathbb{F}) \tag{2.5}$$

is called upper triangular if $b_i^j = 0$ for all $i < j$.

Proposition 2.10 (Schur's Theorem)**.** *If A is a linear operator in a finite-dimensional vector space V and its characteristic polynomial splits, then there exists a basis of V in which the matrix of A is upper triangular.*

In the proof of Schur's theorem, we can recall the result about the characteristic polynomials of a linear operator and its restriction to an invariant subspace from the previous lecture.

3. Primary Decomposition Theorem

Definition 3.1. Let λ be an eigenvalue of a linear operator A in a vector space V. Define

$$V^\lambda := \{x \in V \mid \exists k \in \mathbb{Z}_+ \text{ such that } (A - \lambda \cdot \mathrm{Id})^k(x) = 0\}. \tag{3.1}$$

V^λ is called a generalized eigenspace of A. It immediately follows that V^λ is a subspace of the vector space V.

Exercise 3.2. Can V^λ coincide with V_λ?

Proposition 3.3. *Let λ be an eigenvalue of a linear operator A in a vector space V. We have the following properties:*
a) $V_\lambda \subset V^\lambda$;
b) The subspace V^λ is A-invariant.

Proposition 3.4. *Suppose that A is a linear operator in a finite-dimensional vector space V and $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of its characteristic polynomial. If vectors $x_1, \dots, x_k$ with $x_j \in V^{\lambda_j}$ satisfy*

$$\sum_{j=1}^{k} x_j = 0, \tag{3.2}$$

then $x_j = 0$, $\forall 1 \le j \le k$.

The following two lemmas help us prove Proposition 3.4.

Lemma 3.5. *If λ is an eigenvalue of a linear operator A in a vector space V, then, for every non-zero $x \in V^\lambda$ there exists $k \ge 0$ such that $(A - \lambda \mathrm{Id})^k(x)$ is an eigenvector of the linear operator A with eigenvalue λ.*

Lemma 3.6. *Let v be an eigenvector of a linear operator A with eigenvalue λ. Then, for every polynomial $p \in \mathbb{F}[x]$, the element v is an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.*

Suppose that $U_1, \dots, U_k$ are subspaces of a vector space V. Recall that, by definition,

$$\left(V = \bigoplus_{j=1}^{k} U_j\right) \quad \Longleftrightarrow \quad \left(\forall v \in V \exists! (x_1, \dots, x_k) \text{ such that } x_j \in U_j,\ \forall 1 \le j \le k \text{ and } \sum_{j=1}^{k} x_j = v\right). \tag{3.3}$$

Now we are ready to formulate the main theorem of this lecture.

Proposition 3.7 (Primary Decomposition Theorem)**.** *Suppose that A is a linear operator in a finite-dimensional vector space V and its characteristic polynomial splits:*

$$\chi_A(x) = \prod_{j=1}^{k} (x - \lambda_j)^{m_j}, \tag{3.4}$$

where $\lambda_1, \dots, \lambda_k$ are distinct, $m_j \ge 1$, $\forall 1 \le j \le k$. Then

$$\boxed{V = \bigoplus_{j=1}^{k} V^{\lambda_j};} \tag{3.5}$$

To prove the primary decomposition theorem, one may consider the polynomials

$$q_j := \prod_{1\leq i\leq k,\, i\neq j} (x-\lambda_i)^{m_i} \tag{3.6}$$

and note that

$$\mathrm{im}(q_j(A)) \subseteq V^{\lambda_j}, \quad \forall 1 \leq j \leq k. \tag{3.7}$$

Bézout's lemma for the $GCD(q_1, \ldots, q_k)$ implies part of the theorem.

Another part of this theorem follows from Proposition 3.4.

Note that one may deduce that $\dim V^{\lambda_j} = m_j$, $\forall 1 \leq j \leq k$ from Schur's theorem. We are not going to do it here. These facts and further details will follow from the Jordan canonical form theorem, which we will prove in the next lecture.

4. Minimal Polynomial of a Linear Operator

Definition 4.1. Suppose that A is a linear operator in a finite-dimensional vector space V. Consider the set

$$I(A) := \{p \in \mathbb{F}[x] \mid p \neq 0, p(A) = 0\}. \tag{4.1}$$

One can show that an element of $I(A)$ of minimal degree is determined uniquely up to multiplication by a non-zero scalar. As usual, we will require that its leading coefficient equals 1 to ensure uniqueness. This monic polynomial is called the minimal (annihilating) polynomial for the linear operator A.

Exercise 4.2. Show that the minimal polynomial for a linear operator is indeed well-defined.

Exercise 4.3. Can the minimal polynomial for some linear operator coincide with its characteristic polynomial?

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 5.1, 5.2, 7.1;
- Axler, Sections 5A-5D;
- Roman, pp. 185-190, 192-197.