LECTURE 10. COMPLEXIFICATION. JORDAN FORM FOR THE REAL CASE.

Please treat the proof of each proposition as an exercise.

1. Complexification.

Definition 1.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as another vector space over $\mathbb{R}$. The tensor product $\mathbb{C}\otimes V$ is also a real vector space. We will introduce a complex vector space structure on $\mathbb{C}\otimes V$ by setting

$$\lambda\cdot(z\otimes v):=(\lambda z\otimes v),\quad \forall\lambda,z\in\mathbb{C},\forall v\in V. \tag{1.1}$$

Here, we define the multiplication by a fixed λ on the set of pairs $\mathbb{C}\times V$. Note that it is bilinear, and then we extend this map to the tensor product $\mathbb{C}\otimes V$ using its universal property. We define addition exactly as in the tensor product $\mathbb{C}\otimes V$. The complex vector space that we have obtained is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Let us also set

$$V_0^{\mathbb{C}}:=\{1\otimes v\mid v\in V\}. \tag{1.2}$$

For every subspace U of the vector space V denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}}:=\{z\otimes u\mid z\in\mathbb{C},u\in U\} \tag{1.3}$$

Exercise 1.2. Let V be a real vector space. Show that $\forall v\in V^{\mathbb{C}}\ \exists u,w\in V$ such that

$$v=1\otimes u+i\otimes w. \tag{1.4}$$

Exercise 1.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V<\infty$, then the dimension of $V^{\mathbb{C}}$ over $\mathbb{C}$ equals $\dim V$.

Exercise 1.4. Construct the natural isomorphism α between real vector spaces V and $V_0^{\mathbb{C}}$.

Definition 1.5. Suppose that V is a real vector space and $A\in\mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}}\in\mathrm{End}(V^{\mathbb{C}})$ of this linear operator A by first setting it on $\mathbb{C}\times V$ by

$$A^{\mathbb{C}}(z\otimes v):=z\otimes(Av), \tag{1.5}$$

and then extending it to $V^{\mathbb{C}}$ using the universal property of tensor product.

Exercise 1.6. Show that

$$\forall v\in V_0^{\mathbb{C}}\quad A^{\mathbb{C}}v\in V_0^{\mathbb{C}}. \tag{1.6}$$

Denote the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ by $A_0^{\mathbb{C}}$. Show that

$$\alpha A\alpha^{-1}=A_0^{\mathbb{C}}. \tag{1.7}$$

So we can view $A^{\mathbb{C}}$ as an extension of a copy of A defined on $V_0^{\mathbb{C}}$. We are going to exploit this viewpoint a lot in the next section.

Example 1.7. Suppose that V is a real vector space, $\alpha=\{v_1,\dots,v_n\}$ is a basis of V, $A\in\mathrm{End}(V)$. Then the set $\beta=\{1\otimes v_1,\dots,1\otimes v_n\}$ is a basis of $V^{\mathbb{C}}$ and the corresponding matrices coincide:

$$[A^{\mathbb{C}}]_\beta=[A]_\alpha. \tag{1.8}$$

Proposition 1.8. *Suppose that V is a real vector space and $A\in\mathrm{End}(V)$. Then we get the following equality for the characteristic polynomials*

$$\chi_{A^{\mathbb{C}}}(x)=\chi_A(x),\quad \forall x\in\mathbb{R}. \tag{1.9}$$

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Note that $\chi_A \in \mathbb{R}[x]$. So Proposition 1.8 states that the characteristic polynomial of the linear operator $A^{\mathbb{C}}$ has real coefficients and coincides with χ_A.

Example 1.9. Suppose that V is a complex vector space, $\gamma = \{v_1, \dots, v_n\}$ is a basis of V, $D \in \mathrm{End}(V)$ and

$$[D]_\gamma = (a_k^l + ib_k^l)_{1 \le k,l \le n} = A + iB, \quad A, B \in \mathrm{Mat}(n \times n, \mathbb{R}). \tag{1.10}$$

Now let us consider V as a vector space over $\mathbb{R}$, denote this vector space by $V^{\mathbb{R}}$ to avoid confusion. Then the map D is simultaneously a linear operator on $V^{\mathbb{R}}$; denote it by $D^{\mathbb{R}}$. The set $\delta = \{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a basis of $V^{\mathbb{R}}$. It follows that

$$[D^{\mathbb{R}}]_\delta = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{1.11}$$

Proposition 1.10. *Suppose that V is a real vector space, $\dim V < \infty$. Define* $\mathrm{Conj} \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{1.12}$$

Then:

a) $(\mathrm{Conj})^2 = \mathrm{Id}$*;*

b)

$$\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} \circ \mathrm{Conj} = A^{\mathbb{C}}; \tag{1.13}$$

c)

$$\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ (\lambda \mathrm{Id}) \circ \mathrm{Conj} = \overline{\lambda}\mathrm{Id}; \tag{1.14}$$

d) The operator $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *is a projection onto* $V_0^{\mathbb{C}}$*;*

e) The operator $\frac{1}{2}(\mathrm{Id} - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *is a projection onto* $iV_0^{\mathbb{C}}$*;*

f)

$$\mathrm{im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{im}\left(\frac{1}{2}(\mathrm{Id} - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{1.15}$$

Proposition 1.11. *Suppose that V is a real finite-dimensional vector space, $A \in \mathrm{End}(V)$, and V^λ is a generalized eigenspace of $A^{\mathbb{C}}$. Then* $\mathrm{Conj}(V^\lambda) = V^{\overline{\lambda}}$.

Proposition 1.12. *Suppose that V is a real finite-dimensional vector space and W is a subspace of $V^{\mathbb{C}}$. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{1.16}$$

2. Jordan form for real linear operators.

In this section, we will obtain an analog of Jordan canonical form for the case of real linear operators. We will complexify a real linear operator, get a Jordan basis for its complexification, and then we will show how to obtain from it a Jordan basis for the initial real case.

Example 2.1. Suppose that V is a real vector space, $A \in \mathrm{End}(V)$, $\chi_A(x)$ has no real roots, and

$$\chi_A(x) = x^2 + cx + d, \quad c, d \in \mathbb{R}. \tag{2.1}$$

Let

$$a := -\frac{c}{2}; \quad b := \frac{\sqrt{4d - c^2}}{2}. \tag{2.2}$$

We will construct a basis of V in which the matrix of A has the form

$$R(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{2.3}$$

Consider

$$\chi_{A^{\mathbb{C}}}(x) = \chi_A(x) = x^2 + cx + d. \tag{2.4}$$

It has the roots $a+bi$ and $a-bi$. Hence, there exists an eigenvector $v \in V^{\mathbb{C}}$ corresponding to the eigenvalue $a+bi$. As we know, there exist $u, w \in V$ such that $v = 1 \otimes u + i \otimes w$. Therefore,

$$A^{\mathbb{C}}(1 \otimes u + i \otimes w) = (a+bi)(1 \otimes u + i \otimes w). \tag{2.5}$$

Expanding this equality, we obtain

$$A(u) = au - bw, \tag{2.6}$$

$$A(w) = bu + aw. \tag{2.7}$$

Note that

$$u = \frac{1}{2}(\mathrm{Id} + \mathrm{Conj})(v); \quad w = \frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})(v). \tag{2.8}$$

It follows from equality (2.2) that $b > 0$.

Exercise 2.2. Let $a, b \in \mathbb{R}$. Put $r := \sqrt{a^2+b^2}$. Show that there exists $\varphi \in \mathbb{R}$ such that

$$R(a,b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) & r\sin(\varphi) \\ -r\sin(\varphi) & r\cos(\varphi) \end{pmatrix} = (r \cdot \mathrm{Id}) \cdot \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}. \tag{2.9}$$

Definition 2.3. Suppose that $a, b \in \mathbb{R}$, $b > 0$. Define a real Jordan block $J_k(a,b) \in \mathrm{Mat}(2k \times 2k, \mathbb{R})$ as

$$J_k(a,b) := \begin{pmatrix} R(a,b) & \mathrm{Id} & 0 & \dots & 0 & 0 \\ 0 & R(a,b) & \mathrm{Id} & \dots & 0 & 0 \\ 0 & 0 & R(a,b) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R(a,b) & \mathrm{Id} \\ 0 & 0 & 0 & \dots & 0 & R(a,b). \end{pmatrix} \tag{2.10}$$

Theorem 2.4 (Jordan form for the real case)**.** *Suppose that A is a linear operator in a finite-dimensional real vector space V. There exists a basis α of the vector space V in which the matrix of A has the following block-diagonal form*

$$[A]_\alpha = \begin{pmatrix} J_{s_1}(\mu_1) & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{s_2}(\mu_2) & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{s_p}(\mu_p) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{k_1}(a_1,b_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{k_2}(a_2,b_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{k_q}(a_q,b_q) \end{pmatrix} \tag{2.11}$$

for some, not necessarily distinct,

$$s_1, s_2, \dots, s_p, k_1, k_2, \dots, k_q \in \mathbb{Z}_+,\ \mu_1, \mu_2, \dots, \mu_p, a_1, \dots, a_q \in \mathbb{R},\ b_1, \dots, b_q \in \mathbb{R}_+. \tag{2.12}$$

The sequence of Jordan blocks is determined uniquely up to a permutation of the blocks.

Proof. Consider $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$. Then we may represent $V^{\mathbb{C}}$ as a direct sum of the generalized eigenspaces:

$$V^{\mathbb{C}} = \bigoplus_{t=1}^{k} V^{\lambda_t} \tag{2.13}$$

Let us pick an arbitrary λ_t and consider the restriction of $A^{\mathbb{C}}$ to the subspace V^{λ_t}, which we will denote by B.

First, assume that $\lambda_t \in \mathbb{R}$. From Proposition 1.12 we get that V^{λ_t} is invariant under the real operator Conj.

Exercise 2.5. Check this claim.

So there exists $U \subset V$ such that

$$V^{\lambda_t} = U^{\mathbb{C}}. \tag{2.14}$$

The restriction of $A - \lambda_t \mathrm{Id}$ to U is nilpotent.

Exercise 2.6. Show this claim.

Hence, we can construct an ordinary Jordan basis for the restriction of $A - \lambda_t \mathrm{Id}$ to U as we did in the proof in the previous lecture.

Now assume that $\lambda_t \notin \mathbb{R}$, $\lambda_t = a + bi$, $a, b \in \mathbb{R}$. For simplicity's sake, let us first consider the case when the JCF for B consists of a single block $J_r(\lambda_t)$ in a basis $v_1, \ldots, v_r$. So we have the following equalities:

$$Bv_1 = \lambda_t v_1, \tag{2.15}$$

$$Bv_2 = \lambda_t v_2 + v_1, \tag{2.16}$$

$$\vdots \qquad \vdots \tag{2.17}$$

$$Bv_r = \lambda_t v_r + v_{r-1}. \tag{2.18}$$

As we know, there exist $u_1, \ldots, u_r, w_1, \ldots, w_r \in V$ such that

$$v_j = 1 \otimes u_j + i \otimes w_j, \quad \forall 1 \leq j \leq r. \tag{2.19}$$

The linear span over $\mathbb{R}$

$$H := \langle u_1, \ldots, u_r, w_1, \ldots, w_r \rangle_{\mathbb{R}} \subset V \tag{2.20}$$

is A-invariant.

Exercise 2.7. Show that the matrix of the restriction of A to the subspace H in the basis $u_1, w_1, u_2, w_2, \ldots, u_r, w_r \in V$ equals $J_r(a, b)$.

Exercise 2.8. Show that

$$V^{\lambda_t} \oplus V^{\overline{\lambda_t}} = (H)^{\mathbb{C}}. \tag{2.21}$$

For completeness, we will briefly describe this construction when the JCF for B does not necessarily consist of one block. Consider a Jordan basis $\{v_{(i|j)} \mid 1 \leq i \leq m_j, 1 \leq j \leq r\}$ for a subspace V^{λ_t}. Suppose it corresponds to the sequence of Jordan blocks $J_{m_1}(\lambda_t), \ldots, J_{m_r}(\lambda_t)$. Then $\{\mathrm{Conj}(v_{(i|j)}) \mid 1 \leq i \leq m_j, 1 \leq j \leq r\}$ is a Jordan basis for the subspace $V^{\overline{\lambda_t}}$. Let us set

$$S_t := \left\{ \frac{1}{2}(\mathrm{Id} + \mathrm{Conj})(v_{(1|1)}), \frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})(v_{(1|1)}), \ldots, \frac{1}{2}(\mathrm{Id} + \mathrm{Conj})(v_{(m_r|r)}), \frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})(v_{(m_r|r)}) \right\} \tag{2.22}$$

Also, define

$$H_t := \langle S_t \rangle. \tag{2.23}$$

Then H_t is A-invariant.

Exercise 2.9. Show that the matrix of the restriction of A to H_t in the basis S_t has the form

$$\begin{pmatrix} J_{m_1}(a,b) & 0 & 0 & 0 \\ 0 & J_{m_2}(a,b) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & J_{m_r}(a,b) \end{pmatrix}. \tag{2.24}$$

We will use the following claim:

Lemma 2.10. *Suppose that $U_1, \dots, U_l$ are subspaces of a real vector space V such that*

$$V^{\mathbb{C}} = \bigoplus_{j=1}^{l} (U_j)^{\mathbb{C}}. \tag{2.25}$$

Then

$$V = \bigoplus_{j=1}^{l} U_j. \tag{2.26}$$

This lemma finishes the proof of the existence of the analogue of a Jordan basis in the real case.

Now let us prove the uniqueness of this Jordan form. Consider the complexification of the restriction of A to the subspace corresponding to a Jordan block $J_{k_j}(a_j, b_j)$. Its JCF consists of two blocks $J_{k_j}(a_j + b_j i)$ and $J_{k_j}(a_j - b_j i)$. The uniqueness of the JCF in the complex case implies the uniqueness in our case.

Exercise 2.11. Explain this implication.

□

Chapters from the textbooks relevant to the lecture:

- Roman, pp.53-55, 82-83.