LECTURE 12. OPERATORS IN AN INNER PRODUCT SPACE.

In this lecture, the notation $\mathbb{F}$ stands for the field $\mathbb{R}$ or $\mathbb{C}$. Please treat every proposition in this lecture as an exercise.

1. Adjoint of an operator

Definition 1.1. Suppose that V is a finite-dimensional inner product space and $A \in \mathrm{End}(V)$. An operator on the vector space V is called the adjoint of A and is denoted by A^* if the following equality holds

$$\langle A^* x, y\rangle = \langle x, Ay\rangle, \quad \forall x, y \in V. \tag{1.1}$$

Exercise 1.2. Let V be a finite-dimensional inner product space. Show that for every $A \in \mathrm{End}(V)$ there exists a unique adjoint $A^* \in \mathrm{End}(V)$. Establish its connection with the adjoint operator on the dual space V^* defined in Lecture 4.

Proposition 1.3. *Let V be a finite-dimensional inner product space. For every $A, B \in \mathrm{End}(V)$*

(1) $(A+B)^* = A^* + B^*$;
(2) $(\lambda A)^* = \overline{\lambda} A^*, \quad \forall \lambda \in \mathbb{F}$;
(3) $(A^*)^* = A$;
(4) $(AB)^* = B^* A^*$;
(5) *if A is invertible then* $(A^{-1})^* = (A^*)^{-1}$.

Proposition 1.4. *Let α be an orthonormal basis of an inner product space V. Then for every $A \in \mathrm{End}(V)$ we have that*

$$[A^*]_\alpha = \overline{([A]_\alpha)^t}. \tag{1.2}$$

Note that if V is a real inner product space, then the equality (1.2) turns into

$$[A^*]_\alpha = ([A]_\alpha)^t. \tag{1.3}$$

Proposition 1.5. *Suppose that V is a finite-dimensional inner product space, U is a subspace of V and $A \in \mathrm{End}(V)$. If the subspace U is A-invariant then its orthogonal complement $U^\perp$ is A^*-invariant.*

Proposition 1.6. *Suppose that V is a finite-dimensional inner product space and $A \in \mathrm{End}(V)$. Then*

(1) $(\ker(A))^\perp = \mathrm{im}(A^*)$;
(2) $(\ker(A^*))^\perp = \mathrm{im}(A)$;
(3)
$$(A \text{ is surjective}) \iff (A^* \text{ is injective}); \tag{1.4}$$
(4)
$$(A \text{ is injective}) \iff (A^* \text{ is surjective}). \tag{1.5}$$

Exercise 1.7. Let V be a finite-dimensional inner product space. Is it true that for every $A \in \mathrm{End}(V)$ we have that

$$\ker(A) = \ker(A^* A) \text{ and } \mathrm{im}(A) = \mathrm{im}(AA^*)? \tag{1.6}$$

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2. Normal operators.

Definition 2.1. Suppose that V is an inner product space. A linear operator $A \in \mathrm{End}(V)$ is called normal if $AA^* = A^*A$.

Exercise 2.2. Find an example of a linear operator that is not normal.

Proposition 2.3. *Suppose that V is a finite-dimensional inner product space and α is an orthonormal basis. Then*

$$(A \in \mathrm{End}(V) \text{ is normal}) \iff \left([A]_\alpha \cdot \overline{([A]_\alpha)^t} = \overline{([A]_\alpha)^t} \cdot [A]_\alpha\right). \tag{2.1}$$

Definition 2.4. Let U be a subspace of an inner product space V. A linear operator $P \in \mathrm{End}(V)$ is called an orthogonal projection if the following conditions hold:

(1) $P^2 = P$;
(2) $P^* = P$.

If $\mathrm{im}P = U$, then the orthogonal projection P is called the orthogonal projection onto U.

Exercise 2.5. Let P be an orthogonal projection. Show that

$$\ker(P) = (\mathrm{im}P)^\perp. \tag{2.2}$$

Exercise 2.6. Show that for every subspace U of a finite-dimensional inner product space V there exists a unique orthogonal projection P_U onto U.

Proposition 2.7. *Suppose that $A \in \mathrm{End}(V)$ is a normal operator. Then*

(1) *$p(A)$ is also normal for every $p \in \mathbb{F}[x]$;*
(2) *$|Ax| = |A^*x| \quad \forall x \in V$;*
(3) *$\ker(A) = \ker(A^*)$;*
(4) *$\ker(A^k) = \ker(A), \quad \forall k \geq 1$;*
(5) *the minimal polynomial for A is a product of distinct irreducible polynomials.*

Proposition 2.8. *Suppose that V is a finite-dimensional complex inner product space, $A \in \mathrm{End}(V)$ is a normal operator. Then A and A^* have a common eigenvector.*

We are ready to formulate a spectral theorem for normal operators in a finite-dimensional complex inner product space. We will add several equivalent ways to state this result.

Proposition 2.9. *Let V be a finite-dimensional complex inner product space. Then the following conditions are equivalent:*

(1) *$A \in \mathrm{End}(V)$ is a normal operator;*
(2) *there exists an orthonormal basis α such that the matrix $[A]_\alpha$ is diagonal;*
(3) *there exists an orthonormal basis of V consisting of eigenvectors of A;*
(4)
$$V = \bigoplus_{j=1}^{k} V_{\lambda_j}, \tag{2.3}$$
where $\lambda_1, \ldots, \lambda_k$ are all eigenvalues of A and V_λ is the eigenspace corresponding to the eigenvalue λ, and the subspaces V_{λ_r} and V_{λ_s} are orthogonal for $r \neq s$;
(5)
$$A = \sum_{j=1}^{k} \lambda_j P_{V_{\lambda_j}}, \tag{2.4}$$
where P_U denotes the orthogonal projection onto a subspace U.

One possible approach to the proof is to use Propositions 2.8 and 1.5. Alternatively, we may use Proposition 2.7.

We are going to consider the real case in the next section.

3. Complexification of an inner product space

Definition 3.1. Let V be a real inner product space. Define an inner product on $V^{\mathbb{C}}$ by

$$\langle 1 \otimes x_1 + i \otimes y_1, 1 \otimes x_2 + i \otimes y_2 \rangle_{\mathbb{C}} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i(\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle). \tag{3.1}$$

Exercise 3.2. Let V be a real inner product space. Show that $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ defines an inner product on the complex vector space $V^{\mathbb{C}}$. We will take it as the default inner product on the complexification $V^{\mathbb{C}}$.

Exercise 3.3. Suppose that V is a finite-dimensional real inner product space, $A \in \mathrm{End}(V)$. Show that

$$(A^{\mathbb{C}})^* = (A^*)^{\mathbb{C}}. \tag{3.2}$$

Proposition 3.4. *Suppose that V is a finite-dimensional real inner product space, $A \in \mathrm{End}(V)$ is a normal operator. As usual, denote by $R(a, b)$ the following matrix*

$$R(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{3.3}$$

Then there exists an orthonormal basis β of V such that

$$[A]_\beta = \begin{pmatrix} \mu_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mu_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R(a_1, b_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R(a_2, b_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R(a_q, b_q) \end{pmatrix} \tag{3.4}$$

for some, not necessarily distinct, $\mu_1, \mu_2, \dots, \mu_p, a_1, \dots, a_q \in \mathbb{R}$, $b_1, \dots, b_q \in \mathbb{R}_+$.

4. Special types of normal operators

Definition 4.1. Let V be a complex inner product space. A linear operator A is called unitary if

$$\langle Ax, Ay \rangle = \langle x, y \rangle, \quad \forall x, y \in V. \tag{4.1}$$

Question 4.2. What can we conclude about the invertibility and the inverse of a unitary operator in a finite-dimensional inner product space?

Proposition 4.3. *Suppose that V is a complex inner product space, $\dim V = n$ and α is an orthonormal basis in V. Then*

$$(A \text{ is unitary}) \iff (\text{The columns (rows) of } [A]_\alpha \text{ form an orthonormal basis in } \mathbb{C}^n). \tag{4.2}$$

Proposition 4.4. *For every unitary operator A in a finite-dimensional complex inner product space V there exists an orthonormal basis α in which the matrix of A is diagonal:*

$$[A]_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \quad \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}, \ |\lambda_j| = 1, 1 \le j \le n. \tag{4.3}$$

Definition 4.5. Let V be a real inner product space. A linear operator A is called orthogonal if

$$\langle Ax, Ay \rangle = \langle x, y \rangle, \quad \forall x, y \in V. \tag{4.4}$$

Proposition 4.6. *Suppose that V is a real inner product space,* $\dim V = n$ *and α is an orthonormal basis in V. Then*

$$(A \text{ is orthogonal}) \iff (\text{The columns (rows) of } [A]_\alpha \text{ form an orthonormal basis in } \mathbb{R}^n). \quad (4.5)$$

Definition 4.7. For $\psi \in \mathbb{R}$ denote by R_ψ the following matrix

$$R_\psi := \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix}. \quad (4.6)$$

Proposition 4.8. *For every orthogonal operator A in a finite-dimensional real inner product space V there exists an orthonormal basis α in which*

$$\boxed{[A]_\alpha = \begin{pmatrix} R_{\psi_1} & 0 & \dots & 0 & 0 & 0 \\ 0 & R_{\psi_2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{\psi_r} & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathrm{Id}_p & 0 \\ 0 & 0 & \dots & 0 & 0 & -\mathrm{Id}_q \end{pmatrix}, \quad p, q, r \geq 0,\ \psi_1, \psi_2, \dots, \psi_r \in \mathbb{R}, \quad (4.7)}$$

and by Id_p we denote the identity $p \times p$ matrix. The sizes of zero matrices in the off-diagonal entries vary depending on the sizes of the corresponding blocks on the main diagonal.

Note that $\dim V = 2r + p + q$ in this case.

Definition 4.9. Let V be a complex inner product space. A linear operator A is called self-adjoint or Hermitian if $A^* = A$.

Proposition 4.10. *For every Hermitian operator A in a finite-dimensional complex inner product space V there exists an orthonormal basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \quad (4.8)$$

Definition 4.11. Let V be a real inner product space. A linear operator A is called symmetric if $A^* = A$.

Proposition 4.12. *For every symmetric operator A in a finite-dimensional real inner product space V there exists an orthonormal basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \quad (4.9)$$

Definition 4.13. Let V be a complex inner product space. A linear operator A is called skew-Hermitian if $A^* = -A$.

Proposition 4.14. *For every skew-Hermitian operator A in a finite-dimensional complex inner product space V there exists an orthonormal basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} i\mu_1 & 0 & \dots & 0 \\ 0 & i\mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & i\mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \quad (4.10)$$

Definition 4.15. Let V be a real inner product space. A linear operator A is called skew-symmetric if $A^* = -A$.

Proposition 4.16. *For every skew-symmetric operator A in a finite-dimensional real inner product space V there exists an orthonormal basis α in which*

$$[A]_\alpha = \begin{pmatrix} R(0,b_1) & 0 & \dots & 0 & 0 \\ 0 & R(0,b_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & R(0,b_r) & 0 \\ 0 & 0 & \dots & 0 & 0_p \end{pmatrix}, \quad p, r \geq 0,\ b_1, b_2, \dots, b_r \in \mathbb{R}, \tag{4.11}$$

and 0_p stands for the zero $p \times p$ matrix.

Chapters from the textbooks relevant to this lecture:

- Friedberg, Insel, Spence, Sections 6.3-6.6;
- Axler, Sections 7A, 7B, 7D;
- Roman, pp. 227-239.