PROBLEM SET 2, DUE FEBRUARY 27

2. Basis, Dimension

Problem 2.1. *Let $(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+m})$ be a list of vectors from a vector space V. Determine whether it is always true that*

$$\langle v_1, \dots, v_k, v_{k+1}, \dots, v_{k+m} \rangle = \langle v_1, \dots, v_k \rangle + \langle v_{k+1}, \dots, v_{k+m} \rangle.$$

Problem 2.2. *Let $(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+m})$ be a list of vectors from a vector space V. Is it always true that*

$$\langle v_1, \dots, v_k, v_{k+1}, \dots, v_{k+m} \rangle = \langle v_1, \dots, v_k \rangle \oplus \langle v_{k+1}, \dots, v_{k+m} \rangle?$$

Problem 2.3. *Consider the vector space $\mathbb{R}_2[x]$. Is the set of vectors*

$$\left\{x^2 - x, x^2 + 2x - 3, -x + 1\right\}$$

linearly independent?

Problem 2.4. *Prove that a set of vectors $\{u, v, w\}$ is linearly independent if and only if the set $\{u + v, v + w, u + w\}$ is linearly independent.*

Problem 2.5. *Is the set*

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

a basis of $\mathbb{R}^4$?

Problem 2.6. *Find a basis for the subspace*

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j = 0 \right\} \subset \mathbb{R}^n.$$

Problem 2.7. *Consider the vector space*

$$\mathbb{R}_{n-1}[x] := \left\{ \sum_{j=0}^{n-1} a_j x^j \mid a_0, \dots, a_{n-1} \in \mathbb{R} \right\}$$

of polynomials of degree not greater than $n - 1$. Assume that $x_1, \dots, x_n$ are n pairwise distinct real numbers. Put

$$q_j(x) := \prod_{1 \le i \le n,\, i \ne j} \frac{x - x_i}{x_j - x_i}.$$

Show that $\{q_1, \dots, q_n\}$ is a basis of $\mathbb{R}_{n-1}[x]$.

Problem 2.8. *Find a basis for the subspace*

$$\{p \in \mathbb{R}_n[x] \mid p(0) = 0, p(1) = 0\} \subset \mathbb{R}_n[x],$$

where $\mathbb{R}_n[x]$ is the space of polynomials with real coefficients of degree not greater than n.

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Problem 2.9. *a) Does there exist a basis* $\{p_1, p_2, p_3, p_4\}$ *of* $\mathbb{R}_3[x]$ *such that none of the polynomials* p_1, p_2, p_3, p_4 *have degree* 3*?*

b) Does there exist a basis $\{p_1, p_2, p_3, p_4\}$ *of* $\mathbb{R}_3[x]$ *such that none of the polynomials* p_1, p_2, p_3, p_4 *have degree* 2*?*

Problem 2.10. *Denote by* $\mathrm{Mat}(n \times n, \mathbb{R})$ *the vector space of real-valued* $n \times n$ *matrices, and by* U *the subspace of symmetric matrices.*

a) Find a basis of U*.*

b) Find a subspace W *such that*

$$\mathrm{Mat}(n \times n, \mathbb{R}) = U \oplus W.$$

Problem 2.11. *Define the unitary Lie algebra* $\mathfrak{u}_n$ *by*

$$\mathfrak{u}_n := \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid A + \bar{A}^t = 0\}.$$

Find $\dim(\mathfrak{u}_n)$ *over* $\mathbb{R}$*.*

Problem 2.12. *Find a subset of the set*

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

that is a basis of $\mathbb{R}^3$*.*

Problem 2.13. *Assume that* U *and* W *are subspaces of a vector space* V*, and*

$$\dim U = k \leq \dim W = l \leq \dim V = n$$

What are the possible values of $\dim(U \cap W)$ *in terms of* k, l *and* n*?*