PROBLEM SET 3, DUE MARCH 6

3. Linear Maps

Problem 3.1. *Does there exist* $f \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ *such that*

$$f\left(\begin{pmatrix}-1\\2\\-1\end{pmatrix}\right) = 1, \quad f\left(\begin{pmatrix}3\\-4\\1\end{pmatrix}\right) = 2, \quad f\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right) = 1?$$

Problem 3.2. *a) Is it true that the image of a linearly independent subset under a monomorphism has to be linearly independent?*

b) Is it true that the image of a linearly independent subset under an epimorphism has to be linearly independent?

Problem 3.3. *Assume that* $x_1, \ldots, x_n$ *are* n *pairwise distinct real numbers. Consider the map* $A : \mathbb{R}_k[x] \to \mathbb{R}^n$ *defined by*

$$A(p) := \begin{pmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{pmatrix}.$$

Find a basis of the quotient space $\mathbb{R}_k[x]/\ker(A)$*. Find* $\mathrm{rk}(A)$*.*

Problem 3.4. *Suppose that* V *and* U *are finite-dimensional spaces. For which subspaces* W *of the space* V *there exist* $A \in \mathrm{Hom}(V, U)$ *such that* $\ker A = W$*?*

Problem 3.5. *Suppose that* V *and* U *are finite dimensional subspaces and* $A \in \mathrm{Hom}(V, U)$*. Does there always exist a subspace* $W \subset V$ *such that* $W \cap \ker(A) = \{0\}$ *and* $A(W) = \mathrm{im}(A)$*?*

Problem 3.6. *Suppose that* V *and* U *are finite dimensional subspaces and* $A \in \mathrm{Hom}(V, U)$*. Show that for every subspace* $W \subset U$ *the preimage* $A^{-1}W$ *is a subspace of* V *and*

$$\dim(A^{-1}W) = \dim(\ker(A)) + \dim(\mathrm{im}(A) \cap W).$$

Problem 3.7. *Show that* $A \in \mathrm{Hom}(V, U)$ *is a monomorphism if and only if* $\exists B \in \mathrm{Hom}(U, V)$ *such that* $BA = \mathrm{Id}$*.*

Problem 3.8. *Show that* $A \in \mathrm{Hom}(V, U)$ *is an epimorphism if and only if* $\exists B \in \mathrm{Hom}(U, V)$ *such that* $AB = \mathrm{Id}$*.*

Problem 3.9. *Does there exist a finite-dimensional vector space* V *and linear operators* $A, B \in \mathrm{End}(V)$ *such that* $AB = \mathrm{Id}$ *and* $A \notin \mathrm{Aut}(V)$*?*

Problem 3.10. *Suppose that* V, U, W *are finite-dimensional vector spaces,* $A \in \mathrm{Hom}(V, U)$*,* $B \in \mathrm{Hom}(U, W)$*. Show that*

$$\dim(\ker(BA)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Problem 3.11. *Suppose that* V, U, W *are finite-dimensional vector spaces,* $A \in \mathrm{Hom}(V, U)$*,* $B \in \mathrm{Hom}(U, W)$*. Show that*

$$\mathrm{rk}(BA)) \leq \min\left(\mathrm{rk}(A), \mathrm{rk}(B)\right).$$

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Problem 3.12. *Let V be a finite-dimensional space over a field $\mathbb{F}$. Assume that $A \in \mathrm{End}(V)$ such that*

$$AB = BA \quad \forall B \in \mathrm{End}(V).$$

Does there always exist $\lambda \in \mathbb{F}$ such that $A = \lambda \mathrm{Id}$?

Problem 3.13. *Suppose that P is a linear operator in a vector space V. Show that $P^2 = P$ if and only if the following two conditions hold:*

(1) $V = \ker P \oplus \mathrm{im} P$;
(2) $Px = x, \quad \forall x \in \mathrm{im} P$.

P is called a projection operator.

Problem 3.14. *Suppose that $A \in \mathrm{End}(\mathbb{R}[x])$ and $\deg(Ap) = \deg(p) - 1$, $\forall p \in \mathbb{R}[x]$ such that $\deg(p) \geq 1$. Is it always true that A is an epimorphism?*