PROBLEM SET 4, DUE MARCH 13

4. Change of Basis

All vector spaces in this problem set are assumed to be finite-dimensional.

Problem 4.1. *Suppose that* $A, B \in \mathrm{Hom}(V, U)$. *Show that*

$$(\mathrm{Ker}A = \mathrm{Ker}B) \iff (\exists C \in \mathrm{GL}(U) \text{ such that } B = CA).$$

Problem 4.2. *Suppose that* $A, B \in \mathrm{Hom}(V, U)$. *Show that*

$$(\mathrm{Im}A = \mathrm{Im}B) \iff (\exists C \in \mathrm{GL}(V) \text{ such that } A = BC).$$

Problem 4.3. *Suppose that* $A \in \mathrm{Hom}(V, U)$ *and* $B \in \mathrm{Hom}(V, W)$. *Show that*

$$(\mathrm{Ker}A \subset \mathrm{Ker}B) \iff (\exists C \in \mathrm{Hom}(U, W) \text{ such that } B = CA).$$

Problem 4.4. *Suppose that* $A, B \in \mathrm{Hom}(V, U)$. *Show that*

$$(\mathrm{Im}A \subset \mathrm{Im}B) \iff (\exists C \in \mathrm{End}(V) \text{ such that } A = BC).$$

Problem 4.5. *Suppose that* $U \subset V$, $p \in \mathrm{Hom}(V, V/U)$ *is a canonical projection map, and* $A \in \mathrm{Hom}(V, W)$. *Show that*

$$(\exists B \in \mathrm{Hom}(V/U, W) \text{ such that } A = B \circ p) \iff (U \subset \mathrm{Ker}A).$$

Problem 4.6. *Let* U *be a subspace of a vector space* V. *Define*

$$S := \{A \in \mathrm{Hom}(V, W) \mid U \subset \mathrm{Ker}A\}.$$

Show that S *is a subspace of* $\mathrm{Hom}(V, W)$ *and express its dimension using dimensions of* U, V *and* W.

Problem 4.7. *Let* $A \in \mathrm{End}(V)$. *Define* $T_A \in \mathrm{End}(\mathrm{End}(V))$ *by*

$$T_A(X) := AX.$$

Find $\dim(\mathrm{Ker}(T_A))$.

Problem 4.8. *Let* A *be an* $m \times n$ *matrix. The linear span of its rows is a subspace of the vector space* $\mathbb{F}^n$, *and the linear span of its columns is a subspace of the vector space* $\mathbb{F}^m$. *Show that the dimensions of those two subspaces are equal.*

Problem 4.9. *Let* U *be a subspace of a finite-dimensional space* V. *Define the annihilator of* U *by*

$$\mathrm{ann}(U) := \{f \in V^* \mid f(u) = 0, \forall u \in U\}.$$

Show that $\mathrm{ann}(U)$ *is a subspace of* V^*. *Express* $\dim(\mathrm{ann}(U))$ *using* $\dim V$ *and* $\dim U$.

Problem 4.10. *Let* $\alpha \in \mathrm{Iso}(V, (V^*)^*)$ *be a canonical isomorphism. Show that for every* $U \subset V$ *we have that*

$$\mathrm{ann}(\mathrm{ann}(U)) = \alpha(U).$$

For that reason the set $\mathrm{ann}(W)$ *for* $W \subset V^*$ *can be considered as a subspace of* V *or* $(V^*)^*$ *depending on our preferences.*

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Problem 4.11. *Is it true that*

$$\operatorname{ann}(U \cap W) = \operatorname{ann}(U) + \operatorname{ann}(W)$$

for every $U, W \subset V$*?*

Problem 4.12. *Suppose that* $U \subset V$*,* $p \in \operatorname{Hom}(V, V/U)$ *is a canonical projection map. Show that* p^* *is an isomorphism between* $(V/U)^*$ *and* $\operatorname{ann}(U)$*.*

Problem 4.13. *Suppose that* A *is a linear operator in a finite-dimensional vector space* V*. Show that*

$$\operatorname{Ker}(A^*) = \operatorname{ann}(\operatorname{Im}(A)).$$

Problem 4.14. *Let* $\alpha \in \operatorname{Iso}(V, (V^*)^*)$ *be a canonical isomorphism. Show that for every* $A \in \operatorname{End}(V)$ *we have that*

$$(A^*)^* = \alpha A \alpha^{-1}.$$