PROBLEM SET 6, DUE APRIL 17

6. Tensor Products

In every problem we consider vector spaces over the same field $\mathbb{F}$.

Problem 6.1. *Construct an isomorphism between $\mathbb{F}\otimes V$ and V that does not depend on the choice of bases.*

Problem 6.2. *Let V and U be vector spaces. Suppose that the set*

$$S := \{v \otimes u \mid v \in V, u \in U\} \tag{6.1}$$

is a subspace of $V \otimes U$. Describe an equivalent condition on the vector spaces V and U.

Definition 6.3. Let V, U and W be vector spaces over the same field $\mathbb{F}$. A map $\alpha : V \times U \to W$ is called bilinear if it satisfies the following conditions:

$$\begin{aligned} &\alpha(ax + bz, y) = a\alpha(x, y) + b\alpha(z, y), \\ &\alpha(x, ay + bt) = a\alpha(x, y) + b\alpha(x, t), \quad \forall x, z \in V, \forall y, t \in U, \forall a, b \in \mathbb{F}. \end{aligned} \tag{6.2}$$

If $W = \mathbb{F}$, then a bilinear map is called a bilinear form. Denote the set of bilinear forms on $V \times U$ by $B(V, U)$.

Problem 6.4. *Show that $B(V, U)$ is a subspace of the vector space of all maps from $V \times U$ to $\mathbb{F}$.*

Problem 6.5. *Construct an isomorphism between $V \otimes U$ and $B(V^*, U^*)$ that does not depend on the choice of bases.*

Problem 6.6. *Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, a set $\{v_1, \dots, v_n\}$ is a basis of V, and a set $\{u_1, \dots, u_m\}$ is a basis of U. For an arbitrary element $x \in V \otimes U$ we write the corresponding linear combination*

$$x = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} a_{ij}(x) v_i \otimes u_j, \quad a_{ij}(x) \in \mathbb{F}.$$

The rank of the element x is, by definition, the rank of the matrix

$$(a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Show that the rank of the element x does not depend on the choice of the bases in V and U.

Problem 6.7. *Let $\alpha : V \times U \to W$ be a bilinear map. Is it always true that $\alpha(V \times U)$ is a subspace of the vector space W?*

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Problem 6.8. *Construct an isomorphism between $V \otimes U$ and $U \otimes V$ that does not depend on the choice of bases.*

Problem 6.9. *Construct an isomorphism between $(V \otimes U) \otimes W$ and $V \otimes (U \otimes W)$ that does not depend on the choice of bases.*

Problem 6.10. *Construct an isomorphism between $(V \otimes U)^*$ and $V^* \otimes U^*$ that does not depend on the choice of bases.*

Problem 6.11. *Does there exist an element of $S^2(V)$ of rank one? Does there exist an element of $\Lambda^2(V)$ of rank one?*

Problem 6.12. *Assume that* $\mathrm{char}(\mathbb{F}) \neq 2$. *Is it always true that*

$$V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V)?$$

Problem 6.13. *Suppose that* $\dim V = n$, $\mathrm{char}(\mathbb{F}) \neq 2$ *and* $\alpha \in \Lambda^n(V) \neq 0$. *Define the map $A_\alpha : \Lambda^{n-1}(V) \to V^*$ by the following rule. For an arbitrary $\beta \in \Lambda^{n-1}(V), x \in V$ their wedge product $\beta \wedge x$ belongs to $\Lambda^n(V)$. Therefore, there exists $c(\alpha, \beta, x) \in \mathbb{F}$ such that*

$$\beta \wedge x = c(\alpha, \beta, x) \cdot \alpha.$$

By definition, put

$$(A_\alpha(\beta))(x) := c(\alpha, \beta, x).$$

Show that A_α is well-defined and $A_\alpha \in \mathrm{Hom}(\Lambda^{n-1}(V), V^*)$.

Problem 6.14. *Show that every permutation $\rho \in S_n$ can be represented as some product of the permutations $(1,2)$, $(2,3), \ldots, (n-1,n)$.*

Problem 6.15. *Show that every permutation $\rho \in S_n$ can be represented as some product of the permutations $(1,2)$ and $(1,2,\ldots,n)$.*