PROBLEM SET 7

7. Characteristic Polynomial, Cayley-Hamilton Theorem

Problem 7.1. *Suppose that $p \in \mathbb{C}[x]$ and $\deg p = m$. Show that the polynomial p has m distinct roots if and only if p and p' have no common roots.*

Problem 7.2. *Suppose that polynomials $p_1, \dots, p_n \in \mathbb{F}[x]$ satisfy the condition*

$$GCD(p_i, p_j) = 1, \quad \forall i \neq j.$$

Consider the polynomials $q_1, \dots, q_n$ defined by

$$q_i := \prod_{j \neq i} p_j, \quad 1 \leq i \leq n.$$

Is it always true that

$$GCD(q_1, \dots, q_n) = 1?$$

Problem 7.3. *Does there exist a field $\mathbb{F}$ and an irreducible polynomial $p \in \mathbb{F}[x]$ such that $\deg(p) > 2$?*

Problem 7.4. *Is the subset*

$$\{a_0 + a_1\sqrt[3]{2} + a_2\sqrt[3]{4} \mid a_0, a_1, a_2 \in \mathbb{Q}\}$$

a subfield of the field $\mathbb{R}$?

Problem 7.5. *Suppose that $\delta : \mathrm{Mat}(n \times n, \mathbb{F}) \to \mathbb{F}$ is a map such that the following conditions hold:*

(1) $\delta(AB) = \delta(A) \cdot \delta(B), \quad \forall A, B \in \mathrm{Mat}(n \times n, \mathbb{F})$;

(2)

$$\delta\left(\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}\right) = a, \quad \forall a \in \mathbb{F}.$$

Show that $\delta = \det$.

Definition 7.6. Let A be a linear operator on a finite-dimensional vector space V. Assume that

$$\chi_A(\lambda) = \lambda^n + \sum_{j=0}^{n-1} b_j \lambda^j.$$

Define the trace $\mathrm{Tr}(A)$ of A to be equal to $-b_{n-1}$.

Note that because the characteristic polynomial of a linear operator does not depend on the choice of a basis, neither does the trace.

Date: April 20, 2026.

Problem 7.7. *Let (a_i^j) be the matrix of a linear operator $A \in \mathrm{End}(V)$ in some basis. Find an expression for $\mathrm{Tr}(A)$ in terms of the matrix entries.*

Problem 7.8. *Show that*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \quad \forall A, B \in \mathrm{End}(V).$$

Problem 7.9. *Suppose that $f \in (\mathrm{End}(V))^*$ and the following conditions hold:*
(1) $f(CAC^{-1}) = f(A)$, $\forall C \in \mathrm{GL}(V), \forall A \in \mathrm{End}(V)$;
(2) $f(\mathrm{Id}) = \dim V$.
Show that $f = \mathrm{Tr}$.

Problem 7.10. *Suppose that we know all coefficients of the characteristic polynomial of a linear operator $A \in \mathrm{GL}(V)$. Find the coefficients of the characteristic polynomial of A^{-1}.*

Problem 7.11. *Does there exist a vector space V with $\dim V > 1$, and a linear operator $A \in \mathrm{End}(V)$ such that there are only two A-invariant subspaces - $\{0\}$ and V?*

Problem 7.12. *Suppose that V is a vector space over a field $\mathbb{F}$, $\lambda \in \mathbb{F}$, $A, B \in \mathrm{End}(V)$, $AB = BA$, and*

$$U = \{x \in V \mid Ax = \lambda x\}.$$

Does U have to be B-invariant?

Problem 7.13. *Suppose that $A \in \mathrm{End}(\mathbb{R}^n)$ and its characteristic polynomial has no real roots. Is it always true that $\det(A) > 0$?*

Problem 7.14. *What is the minimal number k such that there exist $A_1, A_2, \ldots, A_k \in \mathrm{End}(\mathbb{C}^n)$ whose products span the vector space $\mathrm{End}(\mathbb{C}^n)$?*