PROBLEM SET 8

8. Eigenvalues and Eigenvectors. Primary Decomposition Theorem.

Vector spaces are assumed to be finite-dimensional unless stated otherwise.

Problem 8.1. *The Fibonacci sequence is defined recursively as*

$$x_0 := 0,\ x_1 := 1,\ x_{n+1} = x_n + x_{n-1},\ \forall n \geq 1.$$

Find an explicit formula for x_n.

Problem 8.2. *Formulate and prove a criterion of invertibility of a square matrix in terms of the roots of the characteristic polynomial.*

Problem 8.3. *For which positive integers* n *does there exist a linear operator* $A \in \mathrm{End}(\mathbb{R}^n)$ *such that it has no eigenvalues?*

Problem 8.4. *Suppose that* A *is a linear operator in a finite-dimensional vector space* V *and* $\mathrm{rk}(A) = k$. *Show that the characteristic polynomial of* A *has at most* $k+1$ *distinct roots.*

Problem 8.5. *Let* λ *be an eigenvalue of* $A \in \mathrm{End}(V)$. *Is it always true that* λ *is also an eigenvalue of* $A^* \in \mathrm{End}(V^*)$*?*

Problem 8.6. *Does there exist an infinite-dimensional complex vector space* V *and an operator* $A \in \mathrm{End}(V)$ *with no eigenvalues?*

Problem 8.7. *Suppose that* $A \in \mathrm{End}(V)$ *and every non-zero vector of* V *is an eigenvector of* A. *Show that there exists* $\lambda \in \mathbb{F}$ *such that*

$$A = \lambda \mathrm{Id}.$$

Problem 8.8. *Let* V *be a finite-dimensional vector space. Show that*

$$\mathrm{Spec}(AB) = \mathrm{Spec}(BA),\ \forall A, B \in \mathrm{End}(V).$$

Problem 8.9. *Let* $m(x)$ *be the minimal polynomial for* $A \in \mathrm{End}(V)$. *Find a formula for the minimal polynomial of* A^{-1}.

Problem 8.10. *Let* $A \in \mathrm{End}(V)$. *Show that*

$$(A \text{ is invertible}) \iff \left(\mathrm{Id} \in \langle A, A^2, \ldots, A^{\dim V - 1}\rangle\right).$$

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