PROBLEM SET 11

11. Inner product spaces.

Problem 11.1. *Let V be a finite-dimensional vector space. Define an inner product on the vector space* $\mathrm{Mat}(n \times n, \mathbb{C})$ *by*

$$\langle A, B\rangle := \mathrm{Tr}\left(A(\overline{B})^t\right).$$

Verify that this formula defines an inner product and find an orthonormal basis of $\mathrm{Mat}(n \times n, \mathbb{C})$.

Problem 11.2. *Prove the inequality*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq n^2, \quad \forall n \in \mathbb{Z}_+, \forall a_1, a_2, \ldots, a_n \in \mathbb{R}_+.$$

Problem 11.3. *Suppose that $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V and $v_1, \ldots, v_n$ are vectors in V such that*

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}}, \quad \forall 1 \leq k \leq n.$$

Show that the list $v_1, \ldots, v_n$ is a basis of V.

Problem 11.4. *Let A be a linear operator in a complex finite-dimensional inner product space. Must there exist an orthonormal basis in which the matrix of A is upper-triangular?*

Problem 11.5. *Suppose that U and W are subspaces of a finite-dimensional inner product space V. Is it always true that*

$$(U + W)^\perp = U^\perp \cap W^\perp ?$$

Problem 11.6. *Does there exist a polynomial $p \in \mathbb{R}_n[x]$ such that for every $q \in \mathbb{R}_n[x]$ the following equality holds:*

$$\int_{-1}^{1} p(x)q(x)dx = q(0) ?$$

Problem 11.7. *Consider the map $A : V \to V^*$ defined in Proposition 4.2 of Lecture 11. For a covector $f \in V^*$, denote the Riesz vector of f by R_f.*

Suppose that U is a subspace of a finite-dimensional inner product space V, $f \in U^*$, $g \in V^*$ and $f(u) = g(u)$ for all $u \in U$. What can we say about the relation between the corresponding Riesz vectors R_f and R_g?

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