PROBLEM SET 12

12. Operators in an inner product space

Problem 12.1. *Suppose U is a subspace of a finite-dimensional inner product space V and let $A \in \mathrm{End}(V)$. Let P_U denote the orthogonal projection onto U. Show that*

$$(U \text{ and } U^\perp \text{ are } A\text{-invariant}) \iff (AP_U = P_U A).$$

Problem 12.2. *Suppose that V is a finite-dimensional inner product space. Is it true that*

$$\forall A \in \mathrm{End}(V) \quad (A \text{ is normal}) \iff (|Ax| = |A^*x|, \quad \forall x \in V)?$$

Problem 12.3. *Do there exist normal operators A and B such that $A + B$ is not normal?*

Problem 12.4. *Do there exist normal operators A and B such that AB is not normal?*

Problem 12.5. *Let A be an operator in a complex finite-dimensional inner product space V. Show that*

$$(A \text{ is normal}) \iff (\forall B \in \mathrm{End}(V)\ ((AB = BA) \implies (A^*B = BA^*))).$$

Problem 12.6. *Let A be a normal operator in a finite-dimensional complex inner product space V. Determine all $k \in \mathbb{Z}_+$ for which there exists $B_k \in \mathrm{End}(V)$ such that*

$$(B_k)^k = A.$$

Problem 12.7. *Do there exist unitary operators A and B such that $A + B$ is not unitary?*

Problem 12.8. *Suppose that A is an operator in a finite-dimensional inner product space V and $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis in V. Show that the value*

$$\sum_{j=1}^{n} |Ae_j|^2$$

does not depend on the choice of the orthonormal basis.

Problem 12.9. *Let V be a finite-dimensional complex inner product space. Show that for every $A \in \mathrm{End}(V)$ there exists a unique pair of self-adjoint operators B and C such that $A = B + iC$. Also show that*

$$(A \text{ is normal}) \iff (BC = CB).$$

Problem 12.10. *Let V be a finite-dimensional inner product space. What condition on V is equivalent to the set of self-adjoint operators being a subspace of $\mathrm{End}(V)$?*

Problem 12.11. *Let U be a unitary operator in a finite-dimensional complex inner product space V such that the operator $\mathrm{Id} - U$ is invertible. Is it always true that the operator*

$$(\mathrm{Id} + U)(\mathrm{Id} - U)^{-1}$$

is self-adjoint?

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