LECTURE 1. DERIVATIVE OF A FUNCTION

Please treat each proposition in this text as an exercise.

1. Definitions

Definition 1.1. Suppose that $S \subset \mathbb{R}$ and $x_0 \in S \cap S'$. A function $f : S \to \mathbb{R}$ is called differentiable at x_0 iff there exists a limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{1.1}$$

Then this limit is called the derivative of the function f at x_0 and is denoted by $f'(x_0)$.

Exercise 1.2. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := 1$ at an arbitrary point $x_0 \in \mathbb{R}$.

Exercise 1.3. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x$ at an arbitrary point $x_0 \in \mathbb{R}$.

Exercise 1.4. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^2$ at an arbitrary point $x_0 \in \mathbb{R}$.

Exercise 1.5. Find the derivative of the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := \frac{1}{x}$ at an arbitrary point $x_0 \in \mathbb{R} \setminus \{0\}$.

Proposition 1.6. *A function $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S \cap S'$ if and only if $\exists L \in \mathbb{R}$, $\forall \varepsilon > 0\, \exists \delta > 0$ such that*

$$|f(x) - (f(x_0) + L(x - x_0))| \le \varepsilon |x - x_0|, \quad \forall x \in U_\delta(x_0) \cap S. \tag{1.2}$$

Proposition 1.7. *Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be differentiable at $x_0 \in S \cap S'$. Then f is continuous at x_0.*

Definition 1.8. Let $S \subset S' \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is called differentiable (on its domain) iff f is differentiable at every point of S.

Suppose that the domain of a function $f : S \to \mathbb{R}$ contains an isolated point and f is differentiable at every point of $S \cap S'$. There is no unified viewpoint regarding whether to call such functions differentiable on their domain.

2. Algebraic properties of the derivative

Proposition 2.1. *Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function, $x_0 \in S' \cap S$, $\lambda \in \mathbb{R}$, and f is differentiable at x_0. Then the function λf is also differentiable at the point x_0 and*

$$(\lambda f)'(x_0) = \lambda f'(x_0). \tag{2.1}$$

Proposition 2.2. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S' \cap S$, and both f and g are differentiable at x_0. Then the function $f + g$ is also differentiable at the point x_0 and*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0). \tag{2.2}$$

Proposition 2.3. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S' \cap S$, and both f and g are differentiable at x_0. Then the function $f - g$ is also differentiable at the point x_0 and*

$$(f - g)'(x_0) = f'(x_0) - g'(x_0). \tag{2.3}$$

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Proposition 2.4 (The Leibniz rule)**.** *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S' \cap S$, and both f and g are differentiable at x_0. Then the function $f \cdot g$ is also differentiable at the point x_0 and*

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \tag{2.4}$$

Proposition 2.5. *Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function, $x_0 \in S' \cap S$, $f(x) \neq 0, \forall x \in S$, and f is differentiable at x_0. Then the function $\frac{1}{f}$ is also differentiable at the point x_0 and*

$$\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{(f(x_0))^2}. \tag{2.5}$$

Proposition 2.6. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S' \cap S$, $g(x) \neq 0, \forall x \in S$, and both f and g are differentiable at x_0. Then the function $\frac{f}{g}$ is also differentiable at the point x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}. \tag{2.6}$$

Exercise 2.7. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^n, n \in \mathbb{Z}_+$ at an arbitrary point $x_0 \in \mathbb{R}$.

3. Chain Rule and Inverse Function Derivative

Proposition 3.1 (Chain rule)**.** *Suppose that $X, Y \subset \mathbb{R}$, $x_0 \in X' \cap X$, $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are two functions, $f(X) \subset Y$, $f(x_0) \in Y' \cap Y$, f is differentiable at x_0, and g is differentiable at $f(x_0)$. Then $g \circ f : X \to \mathbb{R}$ is also differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0). \tag{3.1}$$

Exercise 3.2. Find the derivative of the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := x^{-n}, n \in \mathbb{Z}_+$ at an arbitrary point $x_0 \in \mathbb{R}$.

Proposition 3.3. *Suppose that $X, Y \subset \mathbb{R}$, $x_0 \in X' \cap X$, $f : X \to \mathbb{R}$ is a function, $f(X) = Y$, $f(x_0) = y_0$, f is a bijection between X and Y, f is differentiable at x_0, $f'(x_0) \neq 0$, and $f^{-1} : Y \to X \subset \mathbb{R}$ is continuous at y_0. Then f^{-1} is differentiable at y_0 and*

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}. \tag{3.2}$$

Relevant chapters from the textbooks for this lecture are:

- Tao, Sections 10.1, 10.4;
- Zorich, Sections 5.1, 5.2;
- Abbott, Sections 5.1, 5.2;
- Rudin, pp. 103–105.