LECTURE 2. THE NUMBER e.

Please treat each proposition in this text as an exercise.

1. Useful Results

Definition 1.1. For arbitrary $n \in \mathbb{N}$ define $n!$ (the factorial of n) by

$$n! := \prod_{j=1}^{n} j. \tag{1.1}$$

Note that $0! = 1$.

For arbitrary $k, n \in \mathbb{N}$ such that $k \leq n$ define the binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}. \tag{1.2}$$

Proposition 1.2 (The binomial expansion)**.** *For arbitrary $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have that*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1.3}$$

Proposition 1.3 (Bernoulli's inequality)**.** *Suppose that $x > -1$ and $k \in \mathbb{N}$. Then*

$$(1+x)^k \geq 1 + kx \tag{1.4}$$

If $x \neq 0$ and $k \geq 2$, then the inequality is strict.

2. Definition of the Number e

Proposition 2.1. *The series*

$$\sum_{k=0}^{\infty} \frac{1}{k!} \tag{2.1}$$

is convergent.

Definition 2.2. Define the number e by the following equality:

$$e := \sum_{k=0}^{\infty} \frac{1}{k!}. \tag{2.2}$$

Proposition 2.3. *The number e is irrational.*

Date: September 20, 2025.

3. Another Definition of the Number e

Lemma 3.1. *For every $n \in \mathbb{Z}_+$ we have*

$$\left(1 - \frac{1}{(n+1)^2}\right)^n > \frac{n+1}{n+2}. \tag{3.1}$$

Proposition 3.2. *The sequence*

$$a_n := \left(1 + \frac{1}{n}\right)^n \tag{3.2}$$

is strictly increasing.

Proposition 3.3. *The sequence*

$$b_n := \left(1 + \frac{1}{n}\right)^{n+1} \tag{3.3}$$

is strictly decreasing.

Proposition 3.4. *The sequences*

$$a_n := \left(1 + \frac{1}{n}\right)^n \text{ and } b_n := \left(1 + \frac{1}{n}\right)^{n+1} \tag{3.4}$$

are convergent and have the same limit.

Proposition 3.5.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e. \tag{3.5}$$

Relevant chapters from the textbooks for this lecture are

- Zorich, Sections 3.1.3c), 3.1.4c).