LECTURE 3. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Please treat each proposition in this text as an exercise.

1. Useful Results

Proposition 1.1. *Suppose that $S \subset \mathbb{R}$ is not upper bounded, $f : S \to \mathbb{R}$ is a function, and there exists*

$$\lim_{x \to +\infty} f(x). \tag{1.1}$$

Define

$$M := \{y \in \mathbb{R} \mid \frac{1}{y} \in S\}, \quad g : M \to \mathbb{R},\ g(y) := \frac{1}{y}, \quad h : M \to \mathbb{R},\ h(y) := f(g(y)). \tag{1.2}$$

Then $0 \in M'$, there exists the limit

$$\lim_{y \to 0^+} h(y) \tag{1.3}$$

and

$$\lim_{y \to 0^+} h(y) = \lim_{x \to +\infty} f(x). \tag{1.4}$$

Proposition 1.2. *Suppose that $S, M \subset \mathbb{R}$, $x_0 \in S \cap S'$, $y_0 \in M'$, $f : S \to \mathbb{R}$, $g : M \to \mathbb{R}$ are two functions, $f(S \setminus \{x_0\}) \subset M \setminus \{y_0\}$, $f(x_0) = y_0$, f is continuous at x_0, and the limit*

$$\lim_{y \to y_0} g(y) \tag{1.5}$$

exists. Then the limit

$$\lim_{x \to x_0} (g \circ f)(x) \tag{1.6}$$

exists as well and

$$\lim_{x \to x_0} (g \circ f)(x) = \lim_{y \to y_0} g(y). \tag{1.7}$$

2. Useful Limits

Lemma 2.1.

$$\lim_{n \to \infty} \left(1 - \frac{1}{n^2}\right)^n = 1 \tag{2.1}$$

Proposition 2.2.

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e. \tag{2.2}$$

Proposition 2.3.

$$\lim_{y \to 0^+} (1 + y)^{\frac{1}{y}} = e. \tag{2.3}$$

Lemma 2.4.

$$\lim_{x \to +\infty} \left(1 - \frac{1}{x^2}\right)^x = 1. \tag{2.4}$$

Lemma 2.5.

$$\lim_{y \to 0^+} \left(1 - y^2\right)^{\frac{1}{y}} = 1. \tag{2.5}$$

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Proposition 2.6.

$$\lim_{y\to 0^-} (1+y)^{\frac{1}{y}} = e. \tag{2.6}$$

Proposition 2.7.

$$\lim_{y\to 0} \frac{\log_e(1+y)}{y} = 1. \tag{2.7}$$

Proposition 2.8.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1. \tag{2.8}$$

3. Derivatives of the Exponential and Logarithmic Functions

Exercise 3.1. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}_+$

$$f(x) := e^x. \tag{3.1}$$

Exercise 3.2. Find the derivative of the function $f : \mathbb{R} \to \mathbb{R}_+$

$$f(x) := a^x, \quad a > 0. \tag{3.2}$$

Exercise 3.3. Find the derivative of the function $f : \mathbb{R}_+ \to \mathbb{R}$

$$f(x) := \log_e x. \tag{3.3}$$

Exercise 3.4. Find the derivative of the function $f : \mathbb{R}_+ \to \mathbb{R}$

$$f(x) := \log_a x, \quad a > 0, a \neq 1. \tag{3.4}$$

Exercise 3.5. Find the derivative of the function $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$f(x) := x^a, \quad a \in \mathbb{R}. \tag{3.5}$$

Exercise 3.6. Find the derivative of the function $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$f(x) := x^x. \tag{3.6}$$

Relevant chapters from the textbooks for this lecture are

- Zorich, Sections 3.2.4b), 5.1.5