LECTURE 4. MEAN VALUE THEOREM

Please treat each proposition in this text as an exercise.

1. Rolle's Theorem

Definition 1.1. Let $f : S \to \mathbb{R}$ be a function, $x_0 \in S$. We say that f has a local maximum at x_0 iff $\exists \delta > 0$ such that $\forall x \in U_\delta(x_0) \cap S$ we have that $f(x) \leq f(x_0)$. We say that f has a local minimum at x_0 iff $\exists \delta > 0$ such that $\forall x \in U_\delta(x_0) \cap S$ we have that $f(x) \geq f(x_0)$. If the function f has a local maximum or a local minimum at x_0, we also describe that as f having a local extremum at x_0.

We say that f has a global maximum at x_0 iff $\forall x \in S$ we have that $f(x) \leq f(x_0)$. Similarly, we say that f has a global minimum at x_0 iff $\forall x \in S$ we have that $f(x) \geq f(x_0)$.

Note that the point where the global maximum of a function f is attained does not have to be unique.

Proposition 1.2 (Fermat)**.** *Suppose that a function $f : (a, b) \to \mathbb{R}$ is differentiable and has a local extremum at $x_0 \in (a, b)$. Then*

$$f'(x_0) = 0. \tag{1.1}$$

Exercise 1.3. Suppose that a function $f : [a, b] \to \mathbb{R}$ is differentiable and has a local extremum at $x_0 \in [a, b]$. Is it always true that

$$f'(x_0) = 0? \tag{1.2}$$

Proposition 1.4 (Rolle's Theorem)**.** *Suppose that a continuous function $f : [a, b] \to \mathbb{R}$ is differentiable at every point of the open interval (a, b) and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that*

$$f'(c) = 0. \tag{1.3}$$

Exercise 1.5. Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that a local maximum of f is not also its global maximum?

2. Darboux's Theorem

Proposition 2.1 (Darboux's Theorem)**.** *Suppose that a function $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(a) < f'(b)$. Then*

$$\forall f'(a) < t < f'(b)\, \exists c \in (a, b) \text{ such that } f'(c) = t. \tag{2.1}$$

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3. Mean Value Theorems

Proposition 3.1 (Mean Value Theorem)**.** *Suppose that a continuous function $f : [a, b] \to \mathbb{R}$ is differentiable at every point of the open interval (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.1}$$

Proposition 3.2 (Generalized Mean Value Theorem)**.** *Suppose that continuous functions $f, g : [a, b] \to \mathbb{R}$ are differentiable at every point of the open interval (a, b). Then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{3.2}$$

4. Monotone functions

Proposition 4.1. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. It is monotonically increasing if and only if*

$$f'(x) \geq 0, \forall x \in (a, b). \tag{4.1}$$

The function f is monotonically decreasing if and only if

$$f'(x) \leq 0, \forall x \in (a, b). \tag{4.2}$$

Proposition 4.2. *A differentiable function $f : (a, b) \to \mathbb{R}$ is constant if and only if*

$$f'(x) = 0, \forall x \in (a, b). \tag{4.3}$$

Proposition 4.3. *Let a function $f : (a, b) \to \mathbb{R}$ be differentiable. If $f'(x) > 0$, $\forall x \in (a, b)$ then f is strictly increasing. If $f'(x) < 0$, $\forall x \in (a, b)$ then f is strictly decreasing.*

Exercise 4.4. Let $f : (a, b) \to \mathbb{R}$ be a strictly increasing, differentiable function. Is it always true that

$$f'(x) > 0, \forall x \in (a, b)? \tag{4.4}$$

Exercise 4.5. Suppose that a continuous function $f : [a, b] \to \mathbb{R}$ is differentiable at every point of the open interval (a, b) and

$$f'(x) \geq 0, \forall x \in (a, b). \tag{4.5}$$

Is it always true that f is monotonically increasing?

Exercise 4.6. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) \geq 0, \forall x \in (a, b). \tag{4.6}$$

Is it always true that f is monotonically increasing?

Relevant chapters from the textbooks for this lecture are

- Abbott, Sections 5.2, 5.3;
- Rudin, pp. 107 – 109;
- Tao, Sections 10.2, 10.3;
- Zorich, Sections 5.3.1, 5.3.2.