LECTURE 5. TAYLOR'S THEOREM

Please treat each proposition in this text as an exercise.

1. Taylor's Polynomial

Definition 1.1. Suppose that $f : S \to \mathbb{R}$ is a function and $S \subset S'$. We will define the derivative of order k of the function f (on the set S) denoted by $f^{(k)}$ recursively:

$$f^{(0)}(x) := f(x), \forall x \in S, \quad f^{(k+1)}(x) := \left(f^{(k)}(x)\right)', \forall x \in S. \tag{1.1}$$

We will also call $f^{(k)}$ k-th derivative of f.

The set of functions that are k times differentiable on a set M is denoted by $D^k(M)$. The set of functions differentiable k times on a set M with continuous k-th derivatives is denoted by $C^k(M)$.

If a function has derivatives of all orders, then it is called smooth. The set of smooth functions on a set M is denoted by $C^\infty(M)$.

Exercise 1.2. Let $k \in \mathbb{Z}_+$. Does there exist $f \in C^k(\mathbb{R})$ such that $f^{(k)}$ is not differentiable at 0?

Definition 1.3. Suppose that $x_0 \in \mathbb{R}$, $\delta > 0$, and $f \in D^k(U_\delta(x_0))$. The polynomial

$$T_{f,x_0,k}(x) := \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \tag{1.2}$$

is called the k-th-order Taylor polynomial of the function f centered at x_0.

Exercise 1.4. Find derivatives of any order of $T_{f,x_0,k}(x)$ at the point x_0.

Exercise 1.5. Find the Taylor polynomial centered at an arbitrary point for the function

$$f(x) = e^x. \tag{1.3}$$

Exercise 1.6. Find the Taylor polynomial centered at an arbitrary point for the function

$$f(x) = a^x, \quad a > 0. \tag{1.4}$$

Exercise 1.7. FFind the Taylor polynomial centered at an arbitrary point for the function

$$f(x) = x^n, \quad n \in \mathbb{N}. \tag{1.5}$$

Exercise 1.8. Find the Taylor polynomial with the center at $x_0 = 0$ of the function

$$f(x) = \frac{1}{1 - x}. \tag{1.6}$$

Exercise 1.9. Find the Taylor polynomial with the center at $x_0 = 0$ of the function

$$f(x) = \ln(1 + x). \tag{1.7}$$

Date: October 25, 2025.

2. Taylor's Theorem

Using the Generalized Mean Value Theorem, we can obtain the following result.

Proposition 2.1. *Suppose that $g \in D^k((a,b))$, $g \in C^{k-1}([a,b])$, and $g^{(j)}(a) = 0$, $\forall 0 \leq j \leq k-1$. Then there exists $y \in (a,b)$ such that*

$$g(b) = \frac{g^{(k)}(y)}{k!}(b-a)^k. \tag{2.1}$$

Proposition 2.2 (Taylor's Theorem)**.** *Suppose that $f \in D^k((a,b))$ and $f \in C^{k-1}([a,b])$. Then there exists $y \in (a,b)$ such that*

$$f(b) = T_{f,a,k-1}(b) + \frac{f^{(k)}(y)}{k!}(b-a)^k. \tag{2.2}$$

Proposition 2.3. *Suppose that $f \in D^{k+1}(U_\delta(x_0))$. Then*

$$\lim_{x \to x_0} \frac{f(x) - T_{f,x_0,k}(x)}{(x-x_0)^k} = 0. \tag{2.3}$$

Proposition 2.4. *Suppose that $f \in D^{k+1}(U_\delta(x_0))$,*

$$p(x) = \sum_{j=0}^{k} a_j x^j \tag{2.4}$$

and

$$\lim_{x \to x_0} \frac{f(x) - p(x)}{(x-x_0)^k} = 0. \tag{2.5}$$

Then

$$p = T_{f,x_0,k}. \tag{2.6}$$

Exercise 2.5. Find the Taylor polynomial with the center at $x_0 = 0$ of the function

$$f(x) = x^3 e^x. \tag{2.7}$$

Exercise 2.6. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} e^{-\frac{1}{|x|}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{2.8}$$

Show that $f \in C^\infty(\mathbb{R})$ and that $f^{(k)}(0) = 0$, $\forall k \in \mathbb{N}$.

Relevant chapters from the textbooks for this lecture are

- Rudin, pp. 110-111;
- Apostol, Section 5.12;
- Zorich, Section 5.3.3.