LECTURE 6. L'HOSPITAL'S RULE

Please treat each proposition in this text as an exercise.

1. Versions of L'Hospital's Rule

To begin with, let us state a case when the intricacies of L'Hospital's Rule proofs are not necessary.

Proposition 1.1. *Suppose that $S \subset \mathbb{R}$, $a \in S \cap S'$, the function $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are differentiable at x_0, $f(x_0) = g(x_0) = 0$, $g'(x_0) \neq 0$. Then there exists $\delta > 0$ such that $g(x) \neq 0$, $\forall x \in \dot{U}_\delta(x_0) \cap S$. Denote the restrictions of the functions f and g to $U_\delta(x_0) \cap S$ by $\widetilde{f}$ and $\widetilde{g}$, respectively. Then the limit*

$$\lim_{x \to x_0} \frac{\widetilde{f}(x)}{\widetilde{g}(x)} \tag{1.1}$$

exists and

$$\lim_{x \to x_0} \frac{\widetilde{f}(x)}{\widetilde{g}(x)} = \frac{f'(x_0)}{g'(x_0)}. \tag{1.2}$$

Now let us formulate two well-known versions of L'Hospital's Rule.

Proposition 1.2. *Suppose that $f, g \in D^1((a,b))$, $f, g \in C([a,b))$, $f(a) = g(a) = 0$, $g'(x) \neq 0 \, \forall x \in (a,b)$, and there exists*

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}. \tag{1.3}$$

Then $g(x) \neq 0$, $\forall x \in (a,b)$, there exists

$$\lim_{x \to a^+} \frac{f(x)}{g(x)}, \tag{1.4}$$

and

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}. \tag{1.5}$$

Proposition 1.3. *Suppose that $f, g \in D^1((a,b))$, $g'(x) \neq 0 \, \forall x \in (a,b)$, $g(x) \neq 0 \, \forall x \in (a,b)$,*

$$\lim_{x \to a^+} g(x) = +\infty, \tag{1.6}$$

and there exists

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}. \tag{1.7}$$

Then there exists

$$\lim_{x \to a^+} \frac{f(x)}{g(x)}, \tag{1.8}$$

and

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}. \tag{1.9}$$

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To formulate other possible versions of L'Hospital's Rule we can choose several conditions:
1) the point where we consider the limit from 6 options:

$$a \in \mathbb{R}, a^+, a^-, -\infty, +\infty, \infty, \tag{1.10}$$

2) the value of

$$\lim \frac{f'(x)}{g'(x)} \tag{1.11}$$

from 3 options:

$$L \in \mathbb{R}, -\infty, +\infty, \tag{1.12}$$

and
3) the value of

$$\lim g(x) \tag{1.13}$$

from 4 options:

$$0(\text{in this case we also assume that} \lim f(x) = 0), -\infty, +\infty, \infty, \tag{1.14}$$

That gives us 72 possible versions. We will add just another two of them below to illustrate that the proofs will be very similar.

Proposition 1.4. *Suppose that* $b < a < c$, $f, g \in D^1((b,a) \cup (a,c))$, $g'(x) \neq 0 \, \forall x \in (b,a) \cup (a,c)$, $g(x) \neq 0 \, \forall x \in (b,a) \cup (a,c)$,

$$\lim_{x \to a} g(x) = \infty, \tag{1.15}$$

and there exists

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty. \tag{1.16}$$

Then there exists

$$\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty. \tag{1.17}$$

Proposition 1.5. *Suppose that* $f, g \in D^1((b, +\infty))$, $g'(x) \neq 0 \, \forall x \in (b, +\infty)$, $g(x) \neq 0 \, \forall x \in (b, +\infty)$,

$$\lim_{x \to +\infty} g(x) = -\infty, \tag{1.18}$$

and there exists

$$\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = +\infty. \tag{1.19}$$

Then there exists

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = +\infty. \tag{1.20}$$

2. Applications

Exercise 2.1. Determine for which pairs $a \in \mathbb{R}^+, b \in \mathbb{R}$ the following limit

$$\lim_{x \to +\infty} \frac{x^b}{a^x} \tag{2.1}$$

exists and find its value.

Exercise 2.2. Determine for which $a \in \mathbb{R}^+$ the following limit

$$\lim_{x \to +\infty} x^a \ln(x) \tag{2.2}$$

exists and find its value.

Exercise 2.3. Determine for which $a \in \mathbb{R}^+$ the following limit

$$\lim_{x \to 0^+} x^a \ln(x) \tag{2.3}$$

exists and find its value.

Exercise 2.4. Find the limit

$$\lim_{x\to+\infty} \frac{(x^2+2)^4(\ln(x))^3}{2^{x^2-x}}. \tag{2.4}$$

Relevant chapters from the textbooks for this lecture are

- Rudin, pp. 109–110;
- Tao, Section 10.5;
- Zorich, Section 5.4.4;
- Abbott, pp.158–160;
- Cummings, Section 7.8.