LECTURE 7. THE RIEMANN INTEGRAL

Please treat each proposition in this text as an exercise.

1. Lower and Upper Sums

Definition 1.1. A partition P of a closed interval $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_{n-1}, x_n$, such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \tag{1.1}$$

The norm of the partition P is defined as

$$d(P) := \max_{1\le j\le n}(x_j - x_{j-1}). \tag{1.2}$$

A partition P_2 of a closed interval $[a, b]$ is called a refinement of a partition P_1 of the same closed interval iff $P_1 \subset P_2$.

For two partitions P and P' of a closed interval $[a, b]$ the union $P \cup P'$ is also a partition of the same closed interval. It is called the common refinement of P and P'.

Definition 1.2. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and

$$P = a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \tag{1.3}$$

is a partition of $[a, b]$. Define

$$m_k := \inf_{x\in[x_{k-1},x_k]} f(x), \quad M_k := \sup_{x\in[x_{k-1},x_k]} f(x), \quad 1 \le k \le n. \tag{1.4}$$

The lower sum of f with respect to P is defined as

$$L(f, P) := \sum_{j=1}^{n} m_j(x_j - x_{j-1}). \tag{1.5}$$

Likewise, the upper sum of f with respect to P is given by

$$U(f, P) := \sum_{j=1}^{n} M_j(x_j - x_{j-1}). \tag{1.6}$$

Proposition 1.3. *Suppose that P_2 is a refinement of P_1 and $f : [a, b] \to \mathbb{R}$ is a bounded function. Then the following inequalities hold:*

$$L(f, P_1) \le L(f, P_2) \le U(f, P_2) \le U(f, P_1). \tag{1.7}$$

Proposition 1.4. *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function,*

$$m := \inf_{x\in[a,b]} f(x), \quad M := \sup_{x\in[a,b]} f(x), \tag{1.8}$$

and a partition P_2 is obtained by adding k points to a partition P_1. Then

$$L(f, P_1) \ge L(f, P_2) - k(M - m)d(P_1), \tag{1.9}$$

$$U(f, P_1) \le U(f, P_2) + k(M - m)d(P_1). \tag{1.10}$$

Definition 1.5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define

$$L(f) := \{L(f, P) \mid P \text{ is a partition of } [a, b]\} \tag{1.11}$$

$$U(f) := \{U(f, P) \mid P \text{ is a partition of } [a, b]\}. \tag{1.12}$$

Proposition 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $\forall x \in L(f), y \in U(f)$ we have $x \le y$.*

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2. Definition of the Riemann Integral

Definition 2.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. From Proposition 1.6 it follows that the set $L(f)$ is bounded from above and the set $U(f)$ is bounded from below. We call the function f Riemann integrable on the closed interval $[a, b]$ iff

$$\sup(\mathrm{L(f)}) = \inf(\mathrm{U(f)}). \tag{2.1}$$

In this case we define the Riemann integral of f on $[a, b]$ to be

$$\int_a^b f(x)dx := \sup(\mathrm{L(f)}). \tag{2.2}$$

Proposition 2.2. *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if $\forall \varepsilon > 0$ there exists a partition P of the closed interval $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon. \tag{2.3}$$

Proposition 2.3. *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if $\exists I \in \mathbb{R}$ such that $\forall \varepsilon > 0$ $\exists \delta > 0$ that satisfies the following condition:*
for every partition P of the closed interval $[a, b]$ with $d(P) < \delta$, the following inequalities

$$I - L(f, P) < \varepsilon \text{ and } U(f, P) - I < \varepsilon \tag{2.4}$$

hold.

Proposition 2.4. *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if $\exists I \in \mathbb{R}$ such that for every sequence of partitions (P_n) satisfying the condition*

$$\lim_{n\to\infty} d(P_n) = 0, \tag{2.5}$$

there exist the limits

$$\lim_{n\to\infty} L(f, P_n) \text{ and } \lim_{n\to\infty} U(f, P_n), \tag{2.6}$$

and the following equalities hold:

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = I. \tag{2.7}$$

Exercise 2.5. Provide an example of a bounded function $f : [a, b] \to \mathbb{R}$ that is not Riemann integrable.

3. Riemann Integrability of Monotonic or Continuous Functions

Proposition 3.1. *Every monotonic function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$.*

Definition 3.2. A function $f : [a, b] \to \mathbb{R}$ is called uniformly continuous on $[a, b]$ iff $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

$$\forall x_1, x_2 \in [a, b] \, (|x_1 - x_2| < \delta) \implies (|f(x_1) - f(x_2)| < \varepsilon) \tag{3.1}$$

Proposition 3.3. *Every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b]$.*

Proposition 3.4. *Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$.*

The relevant textbook chapters for this lecture are

- Rudin, pp. 120–125;
- Tao, Sections 11.1–11.3;
- Zorich, Sections 6.1.2, 6.1.3;
- Abbott, Section 7.2;
- Cummings, Sections 8.3-8.5.