LECTURE 8. RIEMANN INTEGRAL PROPERTIES

Please treat each proposition in this text as an exercise.

1. Algebraic properties

Definition 1.1. Denote the set of Riemann integrable functions on a closed interval $[a, b]$ by $R[a, b]$.

Proposition 1.2. *Let $f \in R[a, b]$, $\alpha \in \mathbb{R}$. Then $\alpha f \in R[a, b]$ and*

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx. \tag{1.1}$$

Proposition 1.3. *Let $f, g \in R[a, b]$. Then $f + g, f - g \in R[a, b]$ and*

$$\int_a^b (f(x)+g(x)dx = \int_a^b f(x)dx+\int_a^b g(x)dx, \quad \int_a^b (f(x)-g(x)dx = \int_a^b f(x)dx-\int_a^b g(x)dx \tag{1.2}$$

Proposition 1.4. *Let $f \in R[a, b]$, $f(x) \geq 0 \, \forall x \in [a, b]$. Then*

$$\int_a^b f(x)dx \geq 0. \tag{1.3}$$

Proposition 1.5. *Let $f, g \in R[a, b]$, $f(x) \geq g(x) \, \forall x \in [a, b]$. Then*

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx. \tag{1.4}$$

Proposition 1.6. *Let $f \in R[a, b]$. Then $|f| \in R[a, b]$ and*

$$\int_a^b |f(x)|dx \geq \left| \int_a^b f(x)dx \right|. \tag{1.5}$$

Proposition 1.7. *Let $f \in R[a, c]$, $b \in (a, c)$. Then $f \in R[a, b]$, $f \in R[b, c]$ and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{1.6}$$

2. Fundamental Theorem of Calculus

Proposition 2.1. *Let $f \in R[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by*

$$F(x) := \int_a^x f(x)dx. \tag{2.1}$$

Then $F \in C([a, b])$. If the function f is continuous at $x_0 \in [a, b]$, then the function F is differentiable at x_0 and

$$F'(x_0) = f(x_0). \tag{2.2}$$

Definition 2.2. Suppose that $S \subset S' \subset \mathbb{R}$ and $f, F : S \to \mathbb{R}$ are functions. F is called an antiderivative of f iff $F \in D^1(S)$ and

$$F'(x) = f(x), \forall x \in S. \tag{2.3}$$

Date: December 5, 2025.

Proposition 2.3 (Fundamental Theorem of Calculus)**.** *Suppose that $f \in R[a,b]$ and $F : [a,b] \to \mathbb{R}$ is an antiderivative of f. Then*

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2.4}$$

Proposition 2.4. *Suppose that $F, G \in D^1([a,b])$, $f := F' \in R[a,b]$, and $g := G' \in R[a,b]$. Then*

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx. \tag{2.5}$$

Definition 2.5. Let $f \in R[a,b]$. Define

$$\int_b^a f(x)dx := -\int_a^b f(x)dx, \int_a^a f(x)dx := 0. \tag{2.6}$$

Proposition 2.6. *Suppose that $g \in C^1([a,b])$, $f \in R[c,d]$ and $g([a,b]) \subset [c,d]$. Then $(f \circ g)g' \in R[a,b]$ and*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy. \tag{2.7}$$

The relevant textbook chapters for this lecture are

- Rudin, pp. 128–134;
- Tao, Sections 11.4, 11.9, 11.10;
- Abbott, Section 7.4, 7.5.