LECTURE 1. LIMIT SUPERIOR AND LIMIT INFERIOR OF A SEQUENCE.

Please consider every proposition in this text to be an exercise.

1. Definitions.

Definition 1.1. Let (a_n) be a bounded sequence. Define new sequences (a_k^+) and (a_l^-) by the formulas

$$a_k^+ := \sup\{a_n \mid n \geq k\}; \tag{1.1}$$

$$a_l^- := \inf\{a_n \mid n \geq l\}. \tag{1.2}$$

We then define the limit superior of the sequence (a_n), denoted

$$\overline{\lim_{n\to\infty}}\, a_n, \tag{1.3}$$

by the formula

$$\overline{\lim_{n\to\infty}}\, a_n := \inf\{a_k^+ \mid k \in \mathbb{Z}_+\}. \tag{1.4}$$

Similarly, we define the limit inferior of the sequence (a_n), denoted

$$\underline{\lim_{n\to\infty}}\, a_n, \tag{1.5}$$

by the formula

$$\underline{\lim_{n\to\infty}}\, a_n := \sup\{a_l^- \mid l \in \mathbb{Z}_+\}. \tag{1.6}$$

Exercise 1.2. Let (a_n) be a bounded sequence. Does the sequence (a_k^+) have to converge? If it converges, do we get the equality

$$\lim_{k\to\infty} a_k^+ = \overline{\lim_{n\to\infty}}\, a_n? \tag{1.7}$$

Exercise 1.3. Let

$$a_n := (-1)^n \left(1 + \frac{1}{n}\right). \tag{1.8}$$

Find $\overline{\lim}_{n\to\infty} a_n$ and $\underline{\lim}_{n\to\infty} a_n$.

Exercise 1.4. Suppose that the sequence (a_n) is not necessarily bounded. Suggest a definition of the cases when we can write that

$$\overline{\lim_{n\to\infty}}\, a_n := +\infty \tag{1.9}$$

or

$$\underline{\lim_{n\to\infty}}\, a_n := -\infty. \tag{1.10}$$

Date: February 12, 2025.

2. Properties of the limit superior and limit inferior.

Proposition 2.1. *Let (a_n) be a bounded sequence. Then*

$$\inf\{\mathrm{a_n} \mid \mathrm{n} \in \mathbb{Z}_+\} \leq \underline{\lim_{\mathrm{n}\to\infty}}\, \mathrm{a_n} \leq \overline{\lim_{\mathrm{n}\to\infty}}\, \mathrm{a_n} \leq \sup\{\mathrm{a_n} \mid \mathrm{n} \in \mathbb{Z}_+\}. \tag{2.1}$$

Proposition 2.2. *Let (a_n) be a bounded sequence. If c is a limit point of (a_n) then*

$$\underline{\lim_{n\to\infty}}\, a_n \leq c \leq \overline{\lim_{n\to\infty}}\, a_n. \tag{2.2}$$

Proposition 2.3. *Let (a_n) be a bounded sequence. Then both*

$$\underline{\lim_{n\to\infty}}\, a_n \tag{2.3}$$

and

$$\overline{\lim_{n\to\infty}}\, a_n \tag{2.4}$$

are limit points of the sequence (a_n).

Proposition 2.4. *A real sequence (a_n) converges to a real number L if and only if it is bounded and*

$$\underline{\lim_{n\to\infty}}\, a_n = \overline{\lim_{n\to\infty}}\, a_n = L. \tag{2.5}$$

Exercise 2.5. Can a bounded divergent sequence have a single limit point?

Exercise 2.6. Does there exist a sequence that has a single limit point 0?

3. Completeness of the reals.

Proposition 3.1. *A sequence (a_n) of real numbers is a Cauchy sequence if and only if it is convergent.*

Chapters from the textbooks relevant to the lecture:

- Tao, Section 6.4,
- Abbott, Sections 2.5, 2.6.