LECTURE 2. REAL EXPONENTIATION, PART 3.

Please consider every proposition in this text to be an exercise.

1. Definitions

Proposition 1.1. *Suppose that $x, a \in \mathbb{R}, x > 1$. The set*

$$\{x^q \mid q \in \mathbb{Q},\, q \le a\} \tag{1.1}$$

is upper-bounded, and the set

$$\{x^q \mid q \in \mathbb{Q},\, q \ge a\} \tag{1.2}$$

is lower-bounded.

Proposition 1.2. *Suppose that $x, a \in \mathbb{R}, x > 1$. Then*

$$\sup\{\mathrm{x^q} \mid \mathrm{q} \in \mathbb{Q},\, \mathrm{q} \le \mathrm{a}\} = \inf\{\mathrm{x^q} \mid \mathrm{q} \in \mathbb{Q},\, \mathrm{q} \ge \mathrm{a}\} \tag{1.3}$$

Proposition 1.3. *Suppose that $x, a \in \mathbb{R}, x > 1$ and (q_n) is a sequence of rational numbers that converges to a. Then the sequence (x^{q_n}) is convergent and*

$$\lim_{n\to\infty} x^{q_n} = \sup\{\mathrm{x^q} \mid \mathrm{q} \in \mathbb{Q},\, \mathrm{q} \le \mathrm{a}\}. \tag{1.4}$$

Exercise 1.4. Formulate and prove analogs of Propositions 1.1 - 1.3 for the case $0 < x < 1$.

Definition 1.5. Suppose that $x, a \in \mathbb{R}, x > 0$ and (q_n) is a sequence of rational numbers that converges to a. We define x raised to the power a as follows

$$x^a := \lim_{n\to\infty} x^{q_n}. \tag{1.5}$$

2. Properties

Proposition 2.1. *Suppose that $x, a, b \in \mathbb{R}, x > 0$. Then*

$$x^a \cdot x^b = x^{a+b}. \tag{2.1}$$

Proposition 2.2. *Suppose that $x, y, a \in \mathbb{R}, x > 0, y > 0$. Then*

$$x^a \cdot y^a = (xy)^a. \tag{2.2}$$

Proposition 2.3. *Suppose that $x, a, b \in \mathbb{R}, x > 0$. Then*

$$(x^a)^b = x^{ab}. \tag{2.3}$$

Proposition 2.4. *Suppose $x > y > 0$ are real numbers. If a is a positive real number, then $x^a > y^a$. If a is a negative real number, then $x^a < y^a$.*

Proposition 2.5. *Suppose $x \in \mathbb{R}, x > 1$ and $a, b \in \mathbb{R}$. Then*

$$(x^a > x^b) \quad \Longleftrightarrow \quad (a > b). \tag{2.4}$$

Chapters from the textbooks relevant to the lecture:

- Tao, Section 6.7,
- Zorich, Section 3.2.2d).

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