LECTURE 3. SERIES, ABSOLUTE CONVERGENCE.

Please consider each proposition in this text as an exercise.

1. Definitions

Definition 1.1. Let (a_n) be a real sequence and $m \in \mathbb{Z}$. An expression

$$\sum_{j=m}^{\infty} a_j \tag{1.1}$$

is called a (formal) series.

Define n-th partial sum of this series as

$$\sum_{j=m}^{n} a_j. \tag{1.2}$$

If the sequence of partial sums

$$\left(\sum_{j=m}^{n} a_j\right) \tag{1.3}$$

is convergent, then we say that the series

$$\sum_{j=m}^{\infty} a_j \tag{1.4}$$

is also convergent and we define

$$\sum_{j=m}^{\infty} a_j := \lim_{n\to\infty} \left(\sum_{j=m}^{n} a_j\right). \tag{1.5}$$

If the sequence of partial sums is divergent, then we say that the series

$$\sum_{j=m}^{\infty} a_j \tag{1.6}$$

is also divergent, and we do not assign any real number value to that series.

Exercise 1.2. Show that the series

$$\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j \tag{1.7}$$

is convergent.

Exercise 1.3. Show that the series

$$\sum_{j=1}^{\infty} (-1)^j \tag{1.8}$$

is divergent.

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Proposition 1.4 (Zero test)**.** *Let*

$$\sum_{j=m}^{\infty} a_j \tag{1.9}$$

be a convergent series. Then

$$\lim_{n\to\infty} a_n = 0. \tag{1.10}$$

2. Absolute convergence

Definition 2.1. We say that a series

$$\sum_{j=m}^{\infty} a_j \tag{2.1}$$

is absolutely convergent iff the series

$$\sum_{j=m}^{\infty} |a_j| \tag{2.2}$$

is convergent.

Proposition 2.2. *Every absolutely convergent series is convergent. Moreover, in this case,*

$$\Big|\sum_{j=m}^{\infty} a_j\Big| \leq \sum_{j=m}^{\infty} |a_j|. \tag{2.3}$$

Definition 2.3. A convergent series is called conditionally convergent if it does not converge absolutely.

3. Basic properties

Proposition 3.1 (Alternating series test)**.** *Let* (a_n) *be a decreasing sequence of non-negative real numbers. Then the series*

$$\sum_{j=1}^{\infty} (-1)^j a_j \tag{3.1}$$

is convergent if and only if the sequence a_n *converges to* 0*.*

Proposition 3.2. *Suppose that the series*

$$\sum_{j=m}^{\infty} a_j \text{ and } \sum_{j=m}^{\infty} b_j \tag{3.2}$$

are convergent, Then, $\forall c, d \in \mathbb{R}$ *the series*

$$\sum_{j=m}^{\infty} (ca_j + db_j) \tag{3.3}$$

is convergent and

$$\sum_{j=m}^{\infty} (ca_j + db_j) = c\sum_{j=m}^{\infty} a_j + d\sum_{j=m}^{\infty} b_j. \tag{3.4}$$

Proposition 3.3. *If the series*

$$\sum_{j=m}^{\infty} a_j \tag{3.5}$$

converges to L, then for every integer k, the series

$$\sum_{i=m+k}^{\infty} a_{i-k} \tag{3.6}$$

also converges to L.

Relevant chapters from the textbooks for this lecture are::

- Tao, Sections 7.1, 7.2,
- Zorich, Section 3.1.4a).