LECTURE 4. NON-NEGATIVE SERIES, CAUCHY CRITERION

Please consider each proposition in this text as an exercise.

1. Non-negative Series

Proposition 1.1. *Let*

$$\sum_{j=m}^{\infty} a_j \tag{1.1}$$

be a series of non-negative real numbers. Then it converges if and only if $\exists c \in \mathbb{R}$ *such that* $\forall n \geq m$

$$\sum_{j=m}^{n} a_j \leq c. \tag{1.2}$$

This proposition allows us to deduce the following result

Proposition 1.2. *Suppose that* $|a_j| \leq b_j$, $\forall j \geq m$ *and the series*

$$\sum_{j=m}^{\infty} b_j \tag{1.3}$$

is convergent. Then the series

$$\sum_{j=m}^{\infty} a_j \tag{1.4}$$

is absolutely convergent and

$$\Big|\sum_{j=m}^{\infty} a_j\Big| \leq \sum_{j=m}^{\infty} |a_j| \leq \sum_{j=m}^{\infty} b_j \tag{1.5}$$

Exercise 1.3. Show that the series

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{2^{((j-10)^2+j)} j^3} \tag{1.6}$$

is absolutely convergent.

2. Cauchy Criterion

Proposition 2.1. *Let* (a_j) *be a decreasing sequence of non-negative numbers. Then the series*

$$\sum_{j=1}^{\infty} a_j \tag{2.1}$$

is convergent if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} \tag{2.2}$$

is convergent.

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Proposition 2.2. *The series*

$$\sum_{j=1}^{\infty} j^{\alpha} \tag{2.3}$$

is convergent if and only if $\alpha < -1$.

Exercise 2.3. Is the series

$$\sum_{j=1}^{\infty} \frac{j^2 - 5j - 10}{(j+10)^3} \tag{2.4}$$

absolutely convergent?

Exercise 2.4. Is the series

$$\sum_{j=1}^{\infty} \frac{(j+1)^4}{j^6} \tag{2.5}$$

absolutely convergent?

Relevant chapters from the textbooks for this lecture are:

- Tao, Section 7.3,
- Zorich, Section 3.1.4a)b).