LECTURE 5. ROOT TEST AND RATIO TEST

Please consider each proposition in this text as an exercise.

1. Root Test

Proposition 1.1. *If $|q| < 1$, then the series*

$$\sum_{j=0}^{\infty} q^j \tag{1.1}$$

is absolutely convergent. If $|q| \geq 1$, then the series

$$\sum_{j=0}^{\infty} q^j \tag{1.2}$$

is divergent.

Proposition 1.2. *Let*

$$\sum_{j=1}^{\infty} a_j \tag{1.3}$$

be a series of real numbers. If

$$\overline{\lim_{j\to\infty}} (|a_j|)^{\frac{1}{j}} > 1, \tag{1.4}$$

then the series

$$\sum_{j=1}^{\infty} a_j \tag{1.5}$$

is divergent.

Note that if

$$\underline{\lim_{j\to\infty}} (|a_j|)^{\frac{1}{j}} > 1, \tag{1.6}$$

then

$$\overline{\lim_{j\to\infty}} (|a_j|)^{\frac{1}{j}} > 1, \tag{1.7}$$

hence,

$$\sum_{j=1}^{\infty} a_j \tag{1.8}$$

is divergent.

Proposition 1.3. *Let*

$$\sum_{j=1}^{\infty} a_j \tag{1.9}$$

be a series of real numbers. If

$$\overline{\lim_{j\to\infty}} (|a_j|)^{\frac{1}{j}} < 1, \tag{1.10}$$

then the series

$$\sum_{j=1}^{\infty} a_j \tag{1.11}$$

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is absolutely convergent.

Exercise 1.4. Find an example of an absolutely convergent series

$$\sum_{j=1}^{\infty} a_j \tag{1.12}$$

such that the sequence (a_j) is convergent and

$$\lim_{j\to\infty} (|a_j|)^{\frac{1}{j}} = 1. \tag{1.13}$$

Exercise 1.5. Find an example of a divergent series

$$\sum_{j=1}^{\infty} a_j \tag{1.14}$$

such that the sequence (a_j) is convergent and

$$\lim_{j\to\infty} (|a_j|)^{\frac{1}{j}} = 1. \tag{1.15}$$

Exercise 1.6. Find an example of a divergent series

$$\sum_{j=1}^{\infty} a_j \tag{1.16}$$

such that

$$\lim_{j\to\infty} (|a_j|)^{\frac{1}{j}} < 1. \tag{1.17}$$

2. Ratio Test

Proposition 2.1. *Let (c_n) be a sequence of positive real numbers. Then the following inequalities hold:*

$$\underline{\lim}_{j\to\infty} \frac{c_{n+1}}{c_n} \leq \underline{\lim}_{j\to\infty} (c_n)^{\frac{1}{n}} \leq \overline{\lim}_{j\to\infty} (c_n)^{\frac{1}{n}} \leq \overline{\lim}_{j\to\infty} \frac{c_{n+1}}{c_n} \tag{2.1}$$

This proposition allows us to deduce the following result

Proposition 2.2 (Ratio Test)**.** *Suppose that*

$$\sum_{j=1}^{\infty} a_j \tag{2.2}$$

is a series of non-zero numbers.

If

$$\overline{\lim}_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} < 1, \tag{2.3}$$

then the series

$$\sum_{j=1}^{\infty} a_j \tag{2.4}$$

is absolutely convergent.

If

$$\underline{\lim}_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} > 1, \tag{2.5}$$

then the series

$$\sum_{j=1}^{\infty} a_j \tag{2.6}$$

is divergent.

Exercise 2.3. Find an example of an absolutely convergent series

$$\sum_{j=1}^{\infty} a_j \tag{2.7}$$

such that the sequence (a_j) is convergent and

$$\lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} = 1. \tag{2.8}$$

Exercise 2.4. Find an example of a divergent series

$$\sum_{j=1}^{\infty} a_j \tag{2.9}$$

such that the sequence (a_j) is convergent and

$$\lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} = 1. \tag{2.10}$$

Exercise 2.5. Show that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1. \tag{2.11}$$

Exercise 2.6. Suppose that $|x| < 1$ and $q \in \mathbb{Z}$. Does the series

$$\sum_{j=1}^{\infty} j^q x^j \tag{2.12}$$

have to be absolutely convergent?

Exercise 2.7. Let $x \in \mathbb{R}$. Is the series

$$\sum_{j=1}^{\infty} \frac{x^j}{j!} \tag{2.13}$$

absolutely convergent?

Relevant chapters from the textbooks for this lecture are:

- Tao, Section 7.5,
- Zorich, Section 3.1.4b).