LECTURE 6. SUBSETS OF THE REAL LINE.

Please consider each proposition in this text as an exercise.

1. Types of points in relation to a set

Definition 1.1. For every $\varepsilon > 0$ and for every $x \in \mathbb{R}$ define an ε-neighborhood of the point x to be the following set:

$$U_\varepsilon(x) := \{y \in \mathbb{R} \mid |y - x| < \varepsilon\} = (x - \varepsilon, x + \varepsilon). \tag{1.1}$$

We also define a deleted ε-neighborhood (also called a punctured ε-neighborhood) of the point x as

$$\dot{U}_\varepsilon(x) := U_\varepsilon(x) \setminus \{x\} = \{y \in \mathbb{R} \mid 0 < |y - x| < \varepsilon\} = (x - \varepsilon, x) \cup (x, x + \varepsilon). \tag{1.2}$$

Definition 1.2. Suppose that $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. The point x is called an adherent point of the set S iff

$$\forall \varepsilon > 0 \text{ we have that } U_\varepsilon(x) \cap S \neq \emptyset. \tag{1.3}$$

Proposition 1.3. *Suppose that $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. The point x is an adherent point of the set S iff there exists a sequence, consisting entirely of elements of the set S, which converges to x.*

Definition 1.4. Suppose that $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. The point x is called a limit point (or a cluster point) of the set S iff x is an adherent point of the set $S \setminus \{x\}$.

Definition 1.5. Suppose that $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. The point x is called an isolated point of the set S iff $x \in S$ and x is not a limit point of the set $S \setminus \{x\}$.

Proposition 1.6. *The set of adherent points of a set S is a disjoint union of the set of limit points of S and the set of isolated points of S.*

Definition 1.7. Suppose that $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. The point x is called an interior point of the set S iff

$$\exists \varepsilon > 0 \text{ such that } U_\varepsilon(x) \subset S. \tag{1.4}$$

Exercise 1.8. Suppose that $S \subset \mathbb{R}$ and $x \in \mathbb{R}$. We have introduced four characteristics of the point x in relation to the set S. For which pairs of characteristics can they occur simultaneously for some S and x? Give an answer for each of the six possible cases.

Exercise 1.9. For each of the following sets, describe its sets of isolated points, limit points, and adherent points:

(1) $A := \emptyset$;
(2) $B := \mathbb{R}$;
(3) $C := \mathbb{Q}$;
(4) $D := \mathbb{R} \setminus \mathbb{Q}$;
(5) $E := \mathbb{Z}$;
(6) $F := (0, 1)$;
(7) $G := [0, 1]$;
(8) $H := (-\infty, 0)$;
(9) $I := \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$;
(10) $J := \bigcup_{n \in \mathbb{Z}_+} (\frac{1}{n+1}, \frac{1}{n})$.

Date: April 25, 2025.

2. Open and closed sets

Definition 2.1. A set $S \subset \mathbb{R}$ is called open if every point of the set S is an interior point of S.

Definition 2.2. A set $S \subset \mathbb{R}$ is called closed if it contains all its adherent points.

Note that the empty set is both open and closed.

Exercise 2.3. Does there exist a set that is neither closed nor open?

Proposition 2.4. *A set $S \subset \mathbb{R}$ is open if and only if the set $\mathbb{R} \setminus S$ is closed.*

Proposition 2.5. *A set $S \subset \mathbb{R}$ is both open and closed if and only if either $S = \emptyset$, or $S = \mathbb{R}$.*

Definition 2.6. The set of all adherent points of a set S is called the closure of the set S and is denoted by $\overline{S}$.

Proposition 2.7. *For every set S, its closure $\overline{S}$ is a closed set.*

Proposition 2.8. *A union of an arbitrary family of open sets is open. An intersection of a finite number of open sets is open.*

Exercise 2.9. Formulate and prove the analog of Proposition 2.8 for closed sets.

Exercise 2.10. Does there exist a family of open sets such that its intersection is not open?

3. Heine-Borel theorem for the real line

Proposition 3.1. *A set S is closed and bounded if and only if for every sequence which takes values only in S, there exists a subsequence converging to an element of S.*

Relevant chapters from the textbooks for this lecture are:

- Tao, Section 9.1,
- Abbott, Section 3.2.