# LECTURE 7. THE LIMIT OF A FUNCTION AT A POINT, PART 1.

Please consider each proposition in this text as an exercise.

## 1. Definitions and examples

**Definition 1.1.** Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and $x_0$ is a limit point of the set $S$. A number $L \in \mathbb{R}$ is called a limit of the function $f$ at the point $x_0$ iff $\forall \varepsilon > 0\, \exists \delta > 0$ such that for all $x \in \dot{U}_\delta(x_0) \cap S$ we have that $f(x) \in U_\varepsilon(L)$. In this case we will write

$$\lim_{x \to x_0} f(x) = L. \tag{1.1}$$

**Definition 1.2.** Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and $x_0$ is a limit point of the set $S$. A number $L \in \mathbb{R}$ is called a limit of the function $f$ at the point $x_0$ iff for every sequence $(a_n)$ of elements of the set $S \setminus \{x_0\}$ converging to the point $x_0$ we have that

$$\lim_{n \to \infty} f(a_n) = L. \tag{1.2}$$

In this case we will write

$$\lim_{x \to x_0} f(x) = L. \tag{1.3}$$

**Proposition 1.3.** *Definitions 1.1 and 1.2 are equivalent.*

**Proposition 1.4.** *Suppose that*

$$\lim_{x \to x_0} f(x) = L_1 \text{ and } \lim_{x \to x_0} f(x) = L_2. \tag{1.4}$$

*Then* $L_1 = L_2$.

**Exercise 1.5.** Let $x_0, c \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := c,\ \forall x \in \mathbb{R}. \tag{1.5}$$

Find

$$\lim_{x \to x_0} g(x), \tag{1.6}$$

if it exists.

**Exercise 1.6.** Let $x_0 \in \mathbb{R}$. Define a function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := x,\ \forall x \in \mathbb{R}. \tag{1.7}$$

Find

$$\lim_{x \to x_0} h(x), \tag{1.8}$$

if it exists.

**Exercise 1.7.** Define the Dirichlet function by

$$D(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{1.9}$$

Check for every $x_0 \in \mathbb{R}$ whether

$$\lim_{x \to x_0} D(x) \tag{1.10}$$

exists.

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**Exercise 1.8.** Define the Thomae function by

$$T(x) := \begin{cases} \frac{1}{q}, & \text{if } x \in \mathbb{Q},\ x = \frac{p}{q},\ p \in \mathbb{Z}, q \in \mathbb{Z}_+,\ GCD(p,q) = 1; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{1.11}$$

Check for every $x_0 \in \mathbb{R}$ whether

$$\lim_{x \to x_0} T(x) \tag{1.12}$$

exists.

## 2. Algebraic Properties

**Proposition 2.1.** *Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function, $x_0$ is a limit point of the set $S$, $\lambda \in \mathbb{R}$, and $f$ has a limit at $x_0$. Then the function $\lambda f$ also has a limit at the point $x_0$ and*

$$\lim_{x \to x_0} (\lambda f(x)) = \lambda \lim_{x \to x_0} f(x). \tag{2.1}$$

**Proposition 2.2.** *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0$ is a limit point of the set $S$, and both $f$ and $g$ have limits at the point $x_0$. Then the function $f + g$ also has a limit at the point $x_0$ and*

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x). \tag{2.2}$$

**Proposition 2.3.** *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0$ is a limit point of the set $S$, and both $f$ and $g$ have limits at the point $x_0$. Then the function $f - g$ also has a limit at the point $x_0$ and*

$$\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x). \tag{2.3}$$

**Proposition 2.4.** *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0$ is a limit point of the set $S$, and both $f$ and $g$ have limits at the point $x_0$. Then the function $f \cdot g$ also has a limit at the point $x_0$ and*

$$\lim_{x \to x_0} (f(x) \cdot g(x)) = \left( \lim_{x \to x_0} f(x) \right) \left( \lim_{x \to x_0} g(x) \right). \tag{2.4}$$

**Proposition 2.5.** *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $g(x) \neq 0 \, \forall x \in S$, $x_0$ is a limit point of the set $S$, both $f$ and $g$ have limits at the point $x_0$, and*

$$\lim_{x \to x_0} g(x) \neq 0. \tag{2.5}$$

*Then the function $\frac{f}{g}$ also has a limit at the point $x_0$ and*

$$\lim_{x \to x_0} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}. \tag{2.6}$$

Relevant chapters from the textbooks for this lecture are:

- Abbott, Section 4.1, 4.2,
- Zorich, 3.2.1, 3.2.2,
- Rudin, pp.83 – 85,
- Tao, Section 9.3.