LECTURE 8. THE LIMIT OF A FUNCTION AT A POINT, PART 2

Please consider each proposition in this text as an exercise.

1. More Properties of the Limit of a Function

Exercise 1.1. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has a limit at every point $x \in \mathbb{R}$. Does f have to be bounded?

Proposition 1.2. *Suppose that a function $f : S \to \mathbb{R}$ has a limit at the point $x_0 \in S'$. Then the function*

$$g(x) := |f(x)|, \quad \forall x \in S. \tag{1.1}$$

also has a limit at x_0 and

$$\lim_{x \to x_0} g(x) = |\lim_{x \to x_0} f(x)|. \tag{1.2}$$

Proposition 1.3. *Suppose that $f, g : S \to \mathbb{R}$ are two functions, $x_0 \in S'$, and there exist limits*

$$\lim_{x \to x_0} f(x) \text{ and } \lim_{x \to x_0} g(x). \tag{1.3}$$

Define

$$h_1(x) := \max(f(x), g(x)),\ h_2(x) := \min(f(x), g(x)), \quad \forall x \in S. \tag{1.4}$$

Then there exist

$$\lim_{x \to x_0} h_1(x) \text{ and } \lim_{x \to x_0} h_2(x), \tag{1.5}$$

and

$$\lim_{x \to x_0} h_1(x) = \max(\lim_{x \to x_0} f(x), \lim_{x \to x_0} g(x)), \tag{1.6}$$

$$\lim_{x \to x_0} h_2(x) = \min(\lim_{x \to x_0} f(x), \lim_{x \to x_0} g(x)). \tag{1.7}$$

Proposition 1.4. *Suppose that $f, g : S \to \mathbb{R}$ are two functions, $f(x) \geq g(x)$, $\forall x \in S$, $x_0 \in S'$, and there exist limits*

$$\lim_{x \to x_0} f(x) \text{ and } \lim_{x \to x_0} g(x). \tag{1.8}$$

Then

$$\lim_{x \to x_0} f(x) \geq \lim_{x \to x_0} g(x). \tag{1.9}$$

Proposition 1.5. *Suppose that $f : S \to \mathbb{R}$ is a function, $M \subset S$ and $x_0 \in M'$. Define the restriction $g : M \to \mathbb{R}$ of the function f to the subset M by*

$$g(x) := f(x), \quad \forall x \in M. \tag{1.10}$$

If there exists a limit

$$\lim_{x \to x_0} f(x), \tag{1.11}$$

then there exists a limit

$$\lim_{x \to x_0} g(x), \tag{1.12}$$

Date: June 6, 2025.

and

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} f(x). \tag{1.13}$$

Exercise 1.6. Suppose that $f : S \to \mathbb{R}$ is a function, $M \subset S$, $x_0 \in M'$, $g : M \to \mathbb{R}$ is a restriction of the function f to the subset M, and there exists a limit

$$\lim_{x\to x_0} g(x). \tag{1.14}$$

Does there have to exist a limit

$$\lim_{x\to x_0} f(x)? \tag{1.15}$$

Proposition 1.7. *Suppose that $f : S \to \mathbb{R}$ is a function, $x_0 \in S'$, $\delta > 0$, $M = S \cap U_\delta(x_0)$, $g : M \to \mathbb{R}$ is a restriction of the function f to the subset M, and there exists a limit*

$$\lim_{x\to x_0} g(x). \tag{1.16}$$

Then there exists a limit

$$\lim_{x\to x_0} f(x), \tag{1.17}$$

and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x). \tag{1.18}$$

2. More Definitions

Exercise 2.1. Give definitions of infinite limits at a point:

$$\lim_{x\to x_0} f(x) = +\infty; \tag{2.1}$$

$$\lim_{x\to x_0} f(x) = -\infty; \tag{2.2}$$

$$\lim_{x\to x_0} f(x) = \infty. \tag{2.3}$$

Exercise 2.2. Give definitions of one-sided limits at a point

$$\lim_{x\to x_0+0} f(x) \text{ and } \lim_{x\to x_0-0} f(x). \tag{2.4}$$

Exercise 2.3. Give definitions of limits at infinity:

$$\lim_{x\to+\infty} f(x), \lim_{x\to-\infty} f(x), \text{ and } \lim_{x\to\infty} f(x). \tag{2.5}$$

Relevant chapters from the textbooks for this lecture are:

- Abbott, Section 4.2,
- Zorich, 3.2.2,
- Tao, Section 9.3.