LECTURE 9. CONTINUOUS FUNCTIONS

Please treat each proposition in this text as an exercise.

1. Definitions and examples

Definition 1.1. Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and $x_0 \in S$. The function f is called continuous at the point x_0 iff either x_0 is an isolated point of the set S, or there exists $\lim_{x\to x_0} f(x)$ and

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{1.1}$$

Definition 1.2. Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and $x_0 \in S$. The function f is called continuous at the point x_0 iff $\forall \varepsilon > 0\, \exists \delta > 0$ such that for all $x \in U_\delta(x_0) \cap S$ we have that $f(x) \in U_\varepsilon(f(x_0))$.

Definition 1.3. Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and $x_0 \in S$. The function f is called continuous at the point x_0 iff for every sequence (a_n) of elements of the set S converging to the point x_0 we have that

$$\lim_{n\to\infty} f(a_n) = f(x_0). \tag{1.2}$$

Proposition 1.4. *Definitions 1.1, 1.2, and 1.3 are equivalent.*

Definition 1.5. Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function. f is called continuous iff it is continuous at x for every $x \in S$.

Exercise 1.6. Let $c \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := c,\ \forall x \in \mathbb{R}. \tag{1.3}$$

Is it continuous?

Exercise 1.7. Define a function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := x,\ \forall x \in \mathbb{R}. \tag{1.4}$$

Is it continuous?

Exercise 1.8. Describe the set of points in which the Dirichlet function is continuous.

Exercise 1.9. Describe the set of points in which the Thomae function is continuous.

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2. Algebraic Properties

Proposition 2.1. *Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function, $x_0 \in S$, $\lambda \in \mathbb{R}$, and f is continuous at x_0. Then the function λf is also continuous at the point x_0.*

Proposition 2.2. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S$, and both f and g are continuous at x_0. Then the function $f + g$ is also continuous at the point x_0.*

Proposition 2.3. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S$, and both f and g are continuous at x_0. Then the function $f - g$ is also continuous at the point x_0.*

Proposition 2.4. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S$, and both f and g are continuous at x_0. Then the function $f \cdot g$ is also continuous at the point x_0.*

Proposition 2.5. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S$, both f and g are continuous at x_0, and $g(x) \neq 0$ $\forall x \in S$. Then the function*

$$\frac{f(x)}{g(x)} \tag{2.1}$$

is also continuous at the point x_0.

Proposition 2.6. *Suppose that $S \subset \mathbb{R}$, $f, g : S \to \mathbb{R}$ are real-valued functions, $x_0 \in S$, and both f and g are continuous at x_0. Then the functions $\max(f, g)$ and $\min(f, g)$ are also continuous at the point x_0.*

Exercise 2.7. Define a function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := |x|, \forall x \in \mathbb{R}. \tag{2.2}$$

Is it continuous?

Proposition 2.8. *Let $a \in \mathbb{R}, a > 0$. Then the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) := a^x$, is continuous.*

Proposition 2.9. *Let $b \in \mathbb{R}$. Then the function $g : (0, \infty) \to \mathbb{R}$, defined by $g(x) := x^b$, is continuous.*

Proposition 2.10. *Suppose that $S \subset \mathbb{R}$, $M \subset \mathbb{R}$, $f : S \to \mathbb{R}$, $g : M \to \mathbb{R}$ are real-valued functions, $f(S) \subset M$, $x_0 \in S$, f is continuous at x_0 and g is continuous at $f(x_0)$. Then the function $g \circ f : S \to \mathbb{R}$ is also continuous at x_0.*

Relevant chapters from the textbooks for this lecture are:

- Tao, Section 9.4,
- Abbott, Section 4.3,
- Zorich, 4.1.1, 4.2.1.
- Rudin, pp.85 - 89.