LECTURE 10. THE MAXIMUM PRINCIPLE

Please treat each proposition in this text as an exercise.

1. Miscellaneous properties of functional limits

Let us formulate a useful criterion for the existence of the limit of a function at a point in terms of the right and the left limits.

Definition 1.1. Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and x_0 is a limit point of the set $S \cap (x_0, +\infty)$. A number $L \in \mathbb{R}$ is called the right limit of the function f at the point x_0 iff $\forall \varepsilon > 0\, \exists \delta > 0$ such that for all $x \in (x_0, x_0 + \delta) \cap S$ we have that $f(x) \in U_\varepsilon(L)$. In this case we will write

$$\lim_{x \to x_0^+} f(x) = L. \tag{1.1}$$

Definition 1.2. Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and x_0 is a limit point of the set $S \cap (-\infty, x_0)$. A number $L \in \mathbb{R}$ is called the left limit of the function f at the point x_0 iff $\forall \varepsilon > 0\, \exists \delta > 0$ such that for all $x \in (x_0 - \delta, x_0) \cap S$ we have that $f(x) \in U_\varepsilon(L)$. In this case we will write

$$\lim_{x \to x_0^-} f(x) = L. \tag{1.2}$$

Proposition 1.3. *Suppose that $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ is a real-valued function and x_0 is simultaneously a limit point of the set $S \cap (-\infty, x_0)$ and of the set $S \cap (x_0, +\infty)$. Then the limit of the function f at the point x_0 exists if and only if both the right and the left limits at x_0 exist and they are equal:*

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x). \tag{1.3}$$

In this case they are equal to the limit of the function f at the point x_0.

Let us also state the squeeze theorem for functional limits.

Proposition 1.4. *Suppose that $S \subset \mathbb{R}$, $f, g, h : S \to \mathbb{R}$ are three real-valued functions, $f(x) \le g(x) \le h(x)$, $\forall x \in S$, x_0 is a limit point of the set S, and*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x). \tag{1.4}$$

Then the limit of the function g at the point x_0 exists and

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x). \tag{1.5}$$

2. The maximum principle

The Bolzano–Weierstrass theorem allows us to deduce the following two important results about continuous functions.

Proposition 2.1. *Suppose that $a < b$ and the function $f : [a, b] \to \mathbb{R}$ is continuous. Then f is bounded.*

Proposition 2.2 (The Maximum Principle)**.** *Suppose that $a < b$ and the function $f : [a, b] \to \mathbb{R}$ is continuous. Then f attains a maximum, i. e. $\exists c \in [a, b]$ such that $\forall x \in [a, b]$ we have that $f(x) \le f(c)$.*

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Exercise 2.3. Is the point c in the Maximum Principle theorem uniquely determined?

Exercise 2.4. Does an analogous Minimum Principle hold true?

Exercise 2.5. Suppose that $a < b$ and the function $f : [a, b) \to \mathbb{R}$ is continuous. Does f have to be bounded?

Exercise 2.6. Suppose that the function $f : [a, +\infty) \to \mathbb{R}$ is continuous and bounded. Does f have to attain its maximum?

Exercise 2.7. Suppose that $a < b$, the function $f : [a, b) \to \mathbb{R}$ is continuous and bounded. Does f have to attain its minimum?

Relevant chapters from the textbooks for this lecture are:

- Tao, Section 9.5, 9.6,
- Abbott, Section 4.4.