LECTURE 11. THE INTERMEDIATE VALUE THEOREM

Please treat each proposition in this text as an exercise.

1. Functional Limits at Infinity

Definition 1.1. Suppose that $S \subset \mathbb{R}$, S is not upper bounded, $f : S \to \mathbb{R}$ is a function. A number $L \in \mathbb{R}$ is called the limit of the function f at $+\infty$ iff $\forall \varepsilon > 0\, \exists c \in \mathbb{R}$ such that for all $x > c$ we have that $f(x) \in U_\varepsilon(L)$. In this case we will write

$$\lim_{x \to +\infty} f(x) = L. \tag{1.1}$$

Definition 1.2. Suppose that $S \subset \mathbb{R}$, S is not lower bounded, $f : S \to \mathbb{R}$ is a function. A number $L \in \mathbb{R}$ is called the limit of the function f at $-\infty$ iff $\forall \varepsilon > 0\, \exists c \in \mathbb{R}$ such that for all $x < c$ we have that $f(x) \in U_\varepsilon(L)$. In this case we will write

$$\lim_{x \to -\infty} f(x) = L. \tag{1.2}$$

Proposition 1.3. *Suppose that $S \subset \mathbb{R}$, S is not upper bounded, $f : S \to \mathbb{R}$ is a real-valued function, and*

$$\lim_{x \to +\infty} f(x) \tag{1.3}$$

exists. Then

$$\lim_{x \to 0^+} f\left(\frac{1}{x}\right) \tag{1.4}$$

also exists and

$$\lim_{x \to 0^+} f\left(\frac{1}{x}\right) = \lim_{x \to +\infty} f(x). \tag{1.5}$$

2. The Intermediate Value Theorem

Proposition 2.1. *Suppose that $a < b$, the function $f : [a, b] \to \mathbb{R}$ is continuous, $y \in \mathbb{R}$, and t lies between $f(a)$ and $f(b)$, i.e., either $f(a) \le t \le f(b)$ or $f(b) \le t \le f(a)$. Then $\exists c \in [a, b]$ such that $f(c) = t$.*

Proposition 2.2. *Suppose that $a < b$ and the function $f : [a, b] \to \mathbb{R}$ is continuous. Then*

$$f([a, b]) = [\inf\{\mathrm{f(x)} \mid \mathrm{x} \in [\mathrm{a}, \mathrm{b}]\}, \sup\{\mathrm{f(x)} \mid \mathrm{x} \in [\mathrm{a}, \mathrm{b}]\}]. \tag{2.1}$$

Proposition 2.3. *Let*

$$p(x) = \sum_{j=0}^{2k+1} a_j x^j \tag{2.2}$$

be a polynomial of odd degree. Then p has a real root.

Date: July 26, 2025.

Exercise 2.4. Suppose that a function $f : [0,1] \to [0,1]$ is continuous. Does there always exist an $a \in [0,1]$ such that $f(a) = a$?

3. Monotone Functions

Definition 3.1. A function $f : S \to \mathbb{R}$ is called strictly monotonically increasing iff $f(x) < f(y)$ for all $x, y \in S$ such that $x < y$. A function $f : S \to \mathbb{R}$ is called strictly monotonically decreasing iff $f(x) > f(y)$ for all $x, y \in S$ such that $x < y$. A function is called strictly monotone if it is strictly monotonically increasing or strictly monotonically decreasing.

Proposition 3.2. *Suppose that $a < b$ and the function $f : [a,b] \to \mathbb{R}$ is strictly monotone. Then f is a bijection between $[a,b]$ and $f([a,b])$. Furthermore, $f([a,b]) = [f(a), f(b)]$ or $f([a,b]) = [f(b), f(a)]$, and the inverse function f^{-1} is also continuous.*

4. Logarithmic Functions

Proposition 4.1. *Let $a > 0, a \neq 1$. Define a function $f : \mathbb{R} \to (0, +\infty)$ by*

$$f(x) := a^x. \tag{4.1}$$

Then f is strictly monotone, continuous and surjective.

Definition 4.2. Let $a > 0$. Define a function $f : \mathbb{R} \to (0, +\infty)$ by

$$f(x) := a^x. \tag{4.2}$$

It is called an exponential function. By Propositions 3.2 and 4.1, if $a \neq 1$, there exists its inverse continuous function $\log_a : (0, +\infty) \to \mathbb{R}$, which is called a logarithmic function (to base a). The number

$$\log_a x \tag{4.3}$$

is called the logarithm of x to base a.

Proposition 4.3. *Suppose that $a, b, c > 0$ and $a \neq 1$. Then*

$$\log_a bc = \log_a b + \log_a c. \tag{4.4}$$

Proposition 4.4. *Suppose that $a, b, c > 0$, $a \neq 1$ and $b \neq 1$. Then*

$$\frac{\log_a c}{\log_a b} = \log_b c. \tag{4.5}$$

Proposition 4.5. *Suppose that $a, b > 0$ and $a \neq 1$. Then*

$$\log_a b^c = c \log_a b, \quad \forall c \in \mathbb{R}. \tag{4.6}$$

Relevant chapters from the textbooks for this lecture are:

- Tao, Sections 9.7, 9.8, 9.10
- Abbott, Section 4.5,
- Zorich, Section 4.2.2.