LECTURE 12. COMPACT SUBSETS OF THE REAL LINE

Please treat each proposition in this text as an exercise.

1. Topological definitions of continuity

Proposition 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The function f is continuous if and only if for every open set $U \subset \mathbb{R}$ its preimage $f^{-1}(U)$ is open.*

Proposition 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The function f is continuous if and only if for every closed set $C \subset \mathbb{R}$ its preimage $f^{-1}(C)$ is closed.*

2. Equivalent definitions of compact sets

We will begin with a classical definition of a compact set.

Definition 2.1. Let $M \subset \mathbb{R}$. A family of subsets of $\mathbb{R}$

$$\{U_\alpha \mid \alpha \in I\} \tag{2.1}$$

is called an open cover (or an open covering) of the set M if the following conditions hold:

(1) U_α is open for every $\alpha \in I$;
(2)

$$M \subset \bigcup_{\alpha \in I} U_\alpha. \tag{2.2}$$

A set M is called compact iff for every open cover

$$\{U_\alpha \mid \alpha \in I\} \tag{2.3}$$

there exists a finite subcover

$$\{U_{\alpha_j} \mid 1 \le j \le k\} \tag{2.4}$$

for some $k \in \mathbb{Z}_+$, such that

$$M \subset \bigcup_{1 \le j \le k} U_{\alpha_j}. \tag{2.5}$$

Proposition 2.2. *A set $M \subset \mathbb{R}$ is compact if and only if it is closed and bounded.*

Note that the Heine-Borel theorem allows us to state another equivalent definition of compact subsets of $\mathbb{R}$. We will collect various equivalent definitions of compact subsets of $\mathbb{R}$ at the end of this section.

Some of them can be formulated in terms of properties of continuous functions..

Proposition 2.3. *A set $M \subset \mathbb{R}$ is compact if and only if every continuous function $f : M \to \mathbb{R}$ is bounded.*

Proposition 2.4. *A set $M \subset \mathbb{R}$ is compact if and only if every bounded continuous function $f : M \to \mathbb{R}$ attains its maximum value.*

Here's a list of equivalent definitions of a compact subset of R. It does not claim to be exhaustive.

Proposition 2.5. *A set M is compact if any of the following equivalent conditions hold:*

A: *For every open cover of M there exists a finite subcover;*
B: *M is closed and bounded;*

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C: *Every sequence of elements of M has a subsequence that converges to an element of M;*
D: *Every infinite subset of M has a limit point, and this limit point belongs to M;*
E: *Every continuous function $f : M \to \mathbb{R}$ is bounded;*
F: *Every bounded continuous function $f : M \to \mathbb{R}$ attains its maximum value.*

Proposition 2.6. *Suppose that $M \subset \mathbb{R}$ is a compact set and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then $f(M)$ is also a compact set.*

Relevant chapters from the textbooks for this lecture are:

- Abbott, Sections 3.3, 4.4,
- Rudin, pp.36–40, 89–92.