LECTURE 13. SEQUENCES IN $\mathbb{R}^M$, OPEN AND CLOSED SUBSETS OF $\mathbb{R}^M$

Please treat each proposition in this text as an exercise.

1. Sequences in $\mathbb{R}^m$

Definition 1.1. Let $m \in \mathbb{Z}_+$. The set of all m-tuples $(a_1, \dots, a_m)$ of real numbers is denoted by $\mathbb{R}^m$.

For any two elements $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ define the distance $d(x, y)$ between them by the formula

$$d(x, y) := \sqrt{\sum_{j=1}^{m} (x_j - y_j)^2}. \tag{1.1}$$

Let $x \in \mathbb{R}^m, r \in \mathbb{R}, r > 0$. Define the open ball $U_r(x)$ with center x and radius r by

$$U_r(x) := \{y \in \mathbb{R}^m \mid d(x, y) < r\}. \tag{1.2}$$

$U_r(x)$ is also called the r-neighborhood of the point x.

We also define the deleted r-neighborhood (also called the punctured r-neighborhood) of the point x as

$$\dot{U}_\varepsilon(x) := U_\varepsilon(x) \setminus \{x\}. \tag{1.3}$$

Proposition 1.2 (Triangle Inequality)**.** *For any $m \in \mathbb{Z}_+$, $x, y, z \in \mathbb{R}^m$ the following inequality holds:*

$$d(x, z) \le d(x, y) + d(y, z). \tag{1.4}$$

Definition 1.3. A sequence of elements of $\mathbb{R}^m$ (a_n) is a map

$$a : \mathbb{Z}_+ \to \mathbb{R}^m, \tag{1.5}$$

where we denote $a(n)$ by a_n.

We say that a sequence (a_n) of elements of $\mathbb{R}^m$ converges to a point $L \in \mathbb{R}^m$ if $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0$, $\exists k \in \mathbb{Z}_+$ such that $\forall n > k$ we have $a_n \in U_\varepsilon(L)$.

Proposition 1.4. *If a sequence of elements of $\mathbb{R}^m$ converges to points L_1 and L_2, then $L_1 = L_2$.*

Definition 1.5. If a sequence (a_n) of elements of $\mathbb{R}^m$ converges to $L \in \mathbb{R}^m$ then the point L is called the limit of the sequence (a_n) and is denoted as

$$L = \lim_{n \to \infty} a_n. \tag{1.6}$$

The sequences of elements of $\mathbb{R}^m$ are divided into two classes of convergent and divergent sequences.

Definition 1.6. A sequence of elements of $\mathbb{R}^m$, (a_n), is called bounded if $\exists r \in \mathbb{R}$ such that $a_n \in U_r((0, \dots, 0))$, $\forall n \in \mathbb{Z}_+$.

Proposition 1.7. *Every convergent sequence of elements of $\mathbb{R}^m$ is bounded.*

Definition 1.8. Let (a_n) be a sequence of elements of $\mathbb{R}^m$. For an arbitrary strictly increasing sequence of positive integers (j_k), let us define a new sequence

$$b_k := a_{j_k}. \tag{1.7}$$

The sequence (b_k) is called a subsequence of the sequence (a_n).

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Definition 1.9. Let (a_n) be a sequence of elements of $\mathbb{R}^m$. An element $c \in \mathbb{R}^m$ is called a limit point of the sequence (a_n) iff there exists a subsequence (b_k) of (a_n) such that

$$\lim_{k\to\infty} b_k = c. \tag{1.8}$$

Definition 1.10. Let (a_n) be a sequence of elements of $\mathbb{R}^m$. An element $c \in \mathbb{R}^m$ is called a limit point of the sequence (a_n) iff $\forall \varepsilon > 0\, \forall p \in \mathbb{Z}_+\, \exists k > p$ such that $a_k \in U_\varepsilon(c)$.

Exercise 1.11. Show that Definition 1.9 and Definition 1.10 are equivalent.

Proposition 1.12 (Bolzano-Weierstrass Theorem for $\mathbb{R}^m$)**.** *Every bounded sequence of elements of $\mathbb{R}^m$ has a limit point.*

2. Open and closed sets

Definition 2.1. Suppose that $S \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$. The point x is called a limit point of the set S iff

$$\forall \varepsilon > 0 \text{ we have that } \dot{U}_\varepsilon(x) \cap S \neq \emptyset. \tag{2.1}$$

The point x is called an isolated point of the set S iff $x \in S$ and x is not a limit point of the set S.

The point x is called an adherent point of the set S iff

$$\forall \varepsilon > 0 \text{ we have that } U_\varepsilon(x) \cap S \neq \emptyset. \tag{2.2}$$

The point x is called an interior point of the set S iff

$$\exists \varepsilon > 0 \text{ such that } U_\varepsilon(x) \subset S. \tag{2.3}$$

Definition 2.2. A set $S \subset \mathbb{R}^m$ is called open if every point of the set S is an interior point of S.

Definition 2.3. A set $S \subset \mathbb{R}^m$ is called closed if S contains all its limit points.

Proposition 2.4. *A set $S \subset \mathbb{R}^m$ is open if and only if the complement $\mathbb{R}^m \setminus S$ is closed.*

Definition 2.5. The set of all adherent points of a set $S \subset \mathbb{R}^m$ is called the closure of the set S and is denoted by $\overline{S}$.

Proposition 2.6. *For every set $S \subset \mathbb{R}^m$, its closure $\overline{S}$ is a closed set.*

Proposition 2.7. *The union of an arbitrary family of open sets is open. The intersection of a finite number of open sets is open.*

Proposition 2.8. *The intersection of an arbitrary family of closed sets is closed. The union of a finite number of closed sets is closed.*

3. Heine-Borel theorem for $\mathbb{R}^m$

Proposition 3.1. *A set $S \subset \mathbb{R}^m$ is closed and bounded if and only if every sequence with values in S has a subsequence that converges to an element of S.*

Relevant chapters from the textbooks for this lecture are:

- Zorich, Sections 7.1.