LECTURE 14. MULTIVARIABLE CONTINUOUS FUNCTIONS

Please consider each proposition in this text as an exercise.

1. Definitions of continuity

Definition 1.1. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function, and x_0 is a limit point of the set S. A number $L \in \mathbb{R}$ is called a limit of the function f at the point x_0 iff $\forall \varepsilon > 0\, \exists \delta > 0$ such that for all $x \in \dot{U}_\delta(x_0)$ we have that $f(x) \in U_\varepsilon(L)$. In this case we will write

$$\lim_{x \to x_0} f(x) = L. \tag{1.1}$$

Definition 1.2. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function, and x_0 is a limit point of the set S. A number $L \in \mathbb{R}$ is called a limit of the function f at the point x_0 iff for every sequence (a_n) of elements of the set $S \setminus \{x_0\}$ converging to the point x_0 it holds that

$$\lim_{n \to \infty} f(a_n) = L. \tag{1.2}$$

In this case we will write

$$\lim_{x \to x_0} f(x) = L. \tag{1.3}$$

Proposition 1.3. *Definitions 1.1 and 1.2 are equivalent.*

Proposition 1.4. *Suppose that*

$$\lim_{x \to x_0} f(x) = L_1 \text{ and } \lim_{x \to x_0} f(x) = L_2. \tag{1.4}$$

Then $L_1 = L_2$.

Definition 1.5. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function, and $x_0 \in S$. The function f is called continuous at the point x_0 iff either x_0 is an isolated point of the set S or there exists $\lim_{x \to x_0} f(x)$ and

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{1.5}$$

Definition 1.6. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function, and $x_0 \in S$. The function f is called continuous at the point x_0 iff $\forall \varepsilon > 0\, \exists \delta > 0$ such that for all $x \in U_\delta(x_0) \cap S$ we have that $f(x) \in U_\varepsilon(f(x_0))$.

Definition 1.7. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function, and $x_0 \in S$. The function f is called continuous at the point x_0 iff for every sequence (a_n) of elements of the set S converging to the point x_0 we have that

$$\lim_{n \to \infty} f(a_n) = f(x_0). \tag{1.6}$$

Proposition 1.8. *Definitions 1.5, 1.6, and 1.7 are equivalent.*

Exercise 1.9. Define $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x, y) := \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases} \tag{1.7}$$

Determine whether h is continuous at the point $(0, 0)$.

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Definition 1.10. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function. f is called continuous iff it is continuous at x for every $x \in S$.

Definition 1.11. Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function. f is called bounded iff the set $f(S) \subset \mathbb{R}$ is bounded.

2. Algebraic Properties

Proposition 2.1. *Suppose that $S \subset \mathbb{R}^m$, $f : S \to \mathbb{R}$ is a real-valued function, x_0 is a limit point of the set S, $\lambda \in \mathbb{R}$, and f has a limit at x_0. Then the function λf also has a limit at the point x_0 and*

$$\lim_{x \to x_0} (\lambda f(x)) = \lambda \lim_{x \to x_0} f(x). \tag{2.1}$$

Proposition 2.2. *Suppose that $S \subset \mathbb{R}^m$, $f, g : S \to \mathbb{R}$ are real-valued functions, x_0 is a limit point of the set S, and both f and g have limits at the point x_0. Then the function $f + g$ also has a limit at the point x_0 and*

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x). \tag{2.2}$$

Proposition 2.3. *Suppose that $S \subset \mathbb{R}^m$, $f, g : S \to \mathbb{R}$ are real-valued functions, x_0 is a limit point of the set S, and both f and g have limits at the point x_0. Then the function $f - g$ also has a limit at the point x_0 and*

$$\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x). \tag{2.3}$$

Proposition 2.4. *Suppose that $S \subset \mathbb{R}^m$, $f, g : S \to \mathbb{R}$ are real-valued functions, x_0 is a limit point of the set S, and both f and g have limits at the point x_0. Then the function $f \cdot g$ also has a limit at the point x_0 and*

$$\lim_{x \to x_0} (f(x) \cdot g(x)) = \left(\lim_{x \to x_0} f(x) \right) \left(\lim_{x \to x_0} g(x) \right). \tag{2.4}$$

Proposition 2.5. *Suppose that $S \subset \mathbb{R}^m$, $f, g : S \to \mathbb{R}$ are real-valued functions, $g(x) \neq 0 \, \forall x \in S$, x_0 is a limit point of the set S, both f and g have limits at the point x_0, and*

$$\lim_{x \to x_0} g(x) \neq 0. \tag{2.5}$$

Then the function $\frac{f}{g}$ also has a limit at the point x_0 and

$$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}. \tag{2.6}$$

3. Compact subsets of $\mathbb{R}^m$

Proposition 3.1. *Let $M \subset \mathbb{R}^m$. Then the following conditions on the set M are equivalent:*

- **A:** *For every open cover of M there exists a finite subcover;*
- **B:** *M is closed and bounded;*
- **C:** *Every sequence of elements of M has a subsequence that converges to an element of M;*
- **D:** *Every infinite subset of M has a limit point, and this limit point belongs to M;*
- **E:** *Every continuous function $f : M \to \mathbb{R}$ is bounded;*
- **F:** *Every bounded continuous function $f : M \to \mathbb{R}$ attains its maximum value.*

If at least one of them holds, then M is called a compact set.

Relevant chapters from the textbooks for this lecture are:

- Zorich, Sections 7.2.