LECTURE 15. HOMEOMORPHISMS. CONNECTEDNESS.

Please treat each proposition in this text as an exercise.

1. Continuity

Definition 1.1. Suppose that $k, m \in \mathbb{Z}_+$ and S is a subset of $\mathbb{R}^m$. Every map $F : S \to \mathbb{R}^k$ can be represented as

$$F(x) = (f_1(x), \dots, f_k(x)), \tag{1.1}$$

where $f_i : S \to \mathbb{R}$ is a multivariable function for every $i \in \mathbb{Z}_+$ such that $1 \le i \le k$.

The map F is called continuous at a point $x_0 \in S$ iff f_i is continuous at x_0 for every $i \in \mathbb{Z}_+$ such that $1 \le i \le k$.

Definition 1.2. A map $F : S \subset \mathbb{R}^m \to \mathbb{R}^k$ is called continuous at a point $x_0 \in S$ iff for every open subset $V \subset \mathbb{R}^k$ containing $F(x_0)$ there exists an open subset $U \subset \mathbb{R}^m$ containing x_0 such that

$$F(S \cap U) \subset V. \tag{1.2}$$

Exercise 1.3. Show that Definitions 1.1 and 1.2 are equivalent.

Definition 1.4. A map $F : S \subset \mathbb{R}^m \to \mathbb{R}^k$ is called continuous iff F is continuous at every point $x \in S$.

Definition 1.2 motivates the following terminology.

Definition 1.5. Let S be a subset of $\mathbb{R}^m$. A set $A \subset S$ is called (relatively) open in S iff there exists an open set $U \subset \mathbb{R}^m$ such that $A = U \cap S$. A set $A \subset S$ is called (relatively) closed in S iff there exists a closed set $C \subset \mathbb{R}^m$ such that $A = C \cap S$.

Proposition 1.6. *A map $F : S \subset \mathbb{R}^m \to \mathbb{R}^k$ is continuous if and only if for every open set $V \subset \mathbb{R}^k$ its preimage $F^{-1}(V)$ is relatively open in S.*

Proposition 1.7. *A map $F : S \subset \mathbb{R}^m \to \mathbb{R}^k$ is continuous if and only if for every closed set $C \subset \mathbb{R}^k$ its preimage $F^{-1}(C)$ is relatively closed in S.*

Proposition 1.8. *Suppose that a map $F : S \subset \mathbb{R}^m \to \mathbb{R}^k$ is continuous at $x_0 \in S$, a map $G : M \subset \mathbb{R}^k \to \mathbb{R}^l$ is continuous at $F(x_0) \in M$, and $F(S) \subset M$. Then the map $G \circ F : S \to \mathbb{R}^l$ is continuous at x_0.*

2. Homeomorphisms

Definition 2.1. A map $F : S \subset \mathbb{R}^m \to \mathbb{R}^k$ is called a homeomorphism (between the sets S and $F(S)$) iff the following conditions hold:

(1) F is a bijection between the sets S and $F(S)$;
(2) F is continuous;
(3) The map F^{-1} is also continuous.

Definition 2.2. Sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^k$ are called homeomorphic iff there exists a homeomorphism between A and B.

Proposition 2.3. *Being homeomorphic is an equivalence relation.*

Exercise 2.4. Do there exist two non-homeomorphic sets A and B such that there exists a bijection between them?

Exercise 2.5. Can a bounded set be homeomorphic to an unbounded one?

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3. Compactness, Connectedness

Proposition 3.1. *Suppose that $M \subset \mathbb{R}^m$ is a compact set and $F : M \to \mathbb{R}^k$ is continuous. Then $F(M) \subset \mathbb{R}^k$ is also a compact subset.*

Definition 3.2. A set $M \subset \mathbb{R}^m$ is called connected iff there do not exist non-empty subsets $A, B \subset M$ such that the following conditions hold:

(1) The sets A and B are relatively open in M;
(2) The set M is a disjoint union of A and B.

Definition 3.3. A set $M \subset \mathbb{R}^m$ is called path connected iff for every pair of points $a, b \in M$ there exists a continuous map $\alpha : [0, 1] \to \mathbb{R}^m$ such that following conditions hold:

(1) $\alpha([0, 1]) \subset M$;
(2) $\alpha(0) = a$;
(3) $\alpha(1) = b$.

Proposition 3.4. *Every path connected set is connected.*

Proposition 3.5. *Every open connected subset of $\mathbb{R}^m$ is path connected.*

Proposition 3.6. *Suppose that $M \subset \mathbb{R}^m$ is a connected set and $F : M \to \mathbb{R}^k$ is continuous. Then $F(M) \subset \mathbb{R}^k$ is also a connected set.*

Proposition 3.7. *Suppose that $M \subset \mathbb{R}^m$ is a path connected set and $F : M \to \mathbb{R}^k$ is continuous. Then $F(M) \subset \mathbb{R}^k$ is also a path connected set.*

Relevant chapters from the textbooks for this lecture are:

- Tao, Analysis II, Sections 2.1-2.4.