PROBLEM SET 1, DUE FEBRUARY 14

1. Limit points.

Problem 1.1. *Show that for every* $k \in \mathbb{Z}_+$

$$\lim_{n\to\infty} n^{\frac{1}{k}} = +\infty.$$

Problem 1.2. *For which* $x \in \mathbb{R}$ *is the sequence* x^n *convergent?*

Problem 1.3. *Show that for every* $x \in \mathbb{R}, x > 0$

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1.$$

Problem 1.4. *Does there exist a sequence whose limit points are exactly* -1, 0, *and* 1*?*

Problem 1.5. *(Challenging) Does there exist a sequence whose set of limit points is equal to*

$$\{\frac{1}{n} \mid n \in \mathbb{Z}_+\}?$$

Problem 1.6. *(Challenging) Does there exist a sequence whose set of limit points is equal to* $\mathbb{Z}$*?*

Problem 1.7. *(Challenging) Does there exist a sequence whose set of limit points is equal to* $[0, 1]$*?*

Problem 1.8. *Suggest a definition when it is natural to call* $+\infty$ *a limit point of a sequence.*

Problem 1.9. *Suggest a definition when it is natural to call* ∞ *a limit point of a sequence.*

Problem 1.10. *Does there exist a sequence that has exactly three limit points:* $-\infty, 0, +\infty$*?*

Problem 1.11. *Does the Nested Intervals Property still hold if we replace closed intervals by open intervals?*

Problem 1.12. *Do there exist sequences* (a_n) *and* (b_n) *such that the following conditions hold:*

(1) $a_n > b_n$, $\forall n \in \mathbb{Z}_+$*;*

(2)
$$\overline{\lim_{n\to\infty}} a_n = \overline{\lim_{n\to\infty}} b_n;$$

(3)
$$\underline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} b_n?$$

Date: February 10, 2025.