PROBLEM SET 2

2. Miscellaneous.

Problem 2.1. *Suppose that $A, B \subset \mathbb{R}$ are non-empty upper-bounded sets. Define the set $A + B$ by*

$$A + B := \{x + y \mid x \in A, y \in B\}. \tag{2.1}$$

Show that the set $A + B$ is also upper-bounded and

$$\sup(\mathrm{A} + \mathrm{B}) = \sup(\mathrm{A}) + \sup(\mathrm{B}). \tag{2.2}$$

Problem 2.2. *Suppose that $A, B \subset \mathbb{R}_+$ are non-empty upper-bounded sets. Define the set $A \cdot B$ by*

$$A \cdot B := \{xy \mid x \in A, y \in B\}. \tag{2.3}$$

Show that the set $A \cdot B$ is also upper-bounded and

$$\sup(\mathrm{A} \cdot \mathrm{B}) = \sup(\mathrm{A}) \cdot \sup(\mathrm{B}). \tag{2.4}$$

Problem 2.3. *Suppose that $A \subset \mathbb{R}_+$ is a non-empty upper-bounded set. Define the set* $\mathrm{inv(A)}$ *by*

$$\mathrm{inv(A)} := \{\mathrm{x}^{-1} \mid \mathrm{x} \in \mathrm{A}\}. \tag{2.5}$$

Show that the set $\mathrm{inv(A)}$ *is lower-bounded and*

$$\inf(\mathrm{inv(A)}) = (\sup(\mathrm{A}))^{-1}. \tag{2.6}$$

Problem 2.4. *Reformulate, whenever possible, each of the following conditions on a sequence (a_n) and a number L in simpler terms:*

(1) $\forall \varepsilon > 0\ \exists m \in \mathbb{Z}_+$ *such that* $\forall n > m$ *we get that* $|a_n - L| < \varepsilon$;
(2) $\forall \varepsilon > 0\ \exists m \in \mathbb{Z}_+$ *such that* $\forall n > m$ *we get that* $|a_n - L| > \varepsilon$;
(3) $\forall \varepsilon > 0\ \exists m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| < \varepsilon$;
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(7) $\forall \varepsilon > 0\ \forall m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| < \varepsilon$;
(8) $\forall \varepsilon > 0\ \forall m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| > \varepsilon$;
(9) $\exists \varepsilon > 0\ \exists m \in \mathbb{Z}_+$ *such that* $\forall n > m$ *we get that* $|a_n - L| < \varepsilon$;
(10) $\exists \varepsilon > 0\ \exists m \in \mathbb{Z}_+$ *such that* $\forall n > m$ *we get that* $|a_n - L| > \varepsilon$;
(11) $\exists \varepsilon > 0\ \exists m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| < \varepsilon$;
(12) $\exists \varepsilon > 0\ \exists m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| > \varepsilon$;
(13) $\exists \varepsilon > 0\ \forall m \in \mathbb{Z}_+, \forall n > m$ *we get that* $|a_n - L| < \varepsilon$;
(14) $\exists \varepsilon > 0\ \forall m \in \mathbb{Z}_+, \forall n > m$ *we get that* $|a_n - L| > \varepsilon$;
(15) $\exists \varepsilon > 0\ \forall m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| < \varepsilon$;
(16) $\exists \varepsilon > 0\ \forall m \in \mathbb{Z}_+, \exists n > m$ *such that* $|a_n - L| > \varepsilon$.

Problem 2.5. *Suppose that* $\lim_{n\to\infty} a_n = L$. *Define*

$$b_n := \frac{\sum_{j=1}^{n} a_j}{n}. \tag{2.7}$$

Does the sequence b_n have to be convergent?

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Problem 2.6. *Define a sequence* (a_n) *recursively:*

$$a_1 := 2, \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right). \tag{2.8}$$

Show that (a_n) *is convergent and find its limit.*