1. Basic Definitions

A group is a set G equipped with an operation * that satisfies the following properties:

1. Closure: For all a, b ∈ G, a * b ∈ G.
2. Associativity: For all a, b, c ∈ G, (a * b) * c = a * (b * c).
3. Identity: There exists an element e ∈ G such that for all a ∈ G, a * e = e * a = a.
4. Invertibility: For each a ∈ G, there exists an element b ∈ G such that a * b = b * a = e.

A group (G, *) is abelian if the operation * is commutative, i.e., for all a, b ∈ G, a * b = b * a.

Examples of groups include:

- The set of integers under addition, (Z, +).
- The set of non-zero complex numbers under multiplication, (C\{0}, ·).
- The set of all permutations of a finite set, (S_n, ∘).

2. Inverse Elements

If (G, *) is a group and a ∈ G, then the inverse of a is denoted by a^{-1} and satisfies the property:

a * a^{-1} = a^{-1} * a = e.

3. Group Homomorphisms

Let (G, *) and (H, ·) be groups. A function φ: G → H is a group homomorphism if for all a, b ∈ G:

φ(a * b) = φ(a) · φ(b).

The set of all group homomorphisms from G to H is denoted by Hom(G, H).

4. Group Isomorphisms

If φ: G → H is a bijective group homomorphism, then φ is an isomorphism and G and H are isomorphic groups.

5. Cayley's Theorem

Every group G is isomorphic to a subgroup of the symmetric group on G, i.e., G is a permutation group.

6. Group Actions

A group G acts on a set X if there is a function G × X → X such that:

- For all g ∈ G and x, y ∈ X, g * (x * y) = (g * x) * y.
- For all x ∈ X, e * x = x, where e is the identity element of G.

The action is said to be transitive if for all x, y ∈ X, there exists g ∈ G such that g * x = y.

7. Normal Subgroups

A subgroup N of a group G is normal if for all g ∈ G and n ∈ N, g * n * g^{-1} ∈ N.

Normal subgroups are used to define quotient groups G/N.

8. Sylow's Theorems

Sylow's Theorems provide information about the number and properties of subgroups of a given order in a finite group.

9. Classification of Finite Simple Groups

The classification theorem states that every finite simple group is either cyclic, alternating, or one of 14 families of groups of Lie type.
2. REPRESENTATIONS AND THEIR INVARIANT SUBSPACES

Definition 2.1. Assume $G$ is a group, $F$ is a field, $V$ is a vector space over $F$. The $F$-linear map

$$T : G \to \text{GL}(V)$$

is called a representation of the group $G$ (or the vector space $V$) over the field $F$. The discussion of a representation $T$ is, by definition, the discussion of the vector space $V$.

Question 2.2. For any given group $G$ and vector space $V$ does there always exist a representation of $G$ in $V$?

Now let us define a sum of two representations. First we recall a construction from linear algebra.

Definition 2.3. Assume $V$ and $W$ are vector spaces over $F$. The Cartesian product $V \times W$ with respect to operations of component-wise addition and multiplication by scalars is called the direct sum of $V$ and $W$.

Definition 2.4. Assume $T : G \to \text{GL}(V)$ and $S : G \to \text{GL}(W)$ are representations over $F$. Then define $T \oplus S : G \to \text{GL}(V \oplus W)$ by

$$(T \oplus S)(g) = (T(g), S(g)), \quad \forall g \in G, g \in G.$$ (2.1)

Assume that we require that a representation is a sum of some representations of smaller dimensions. How should we proceed to find these smaller fields?

Definition 2.5. Assume $V' : G \to \text{GL}(V')$ is a representation. A subspace $U \subset V$ is called invariant with respect to $T$ if

$$T(g)u \in U, \quad \forall g \in G, \quad \forall u \in U.$$ (2.2)

Definition 2.6. Assume $T : G \to \text{GL}(V)$ is a representation. $V$ is called irreducible if it has no nontrivial invariant subspaces.

Definition 2.7. Assume $V' : G \to \text{GL}(V')$ is a representation. $V$ is called simple if it is irreducible if for any nontrivial subspace $W$ there exists another nontrivial subspace $W'$ such that $V = W \oplus W'$.

Then the solution $T(g)$ should be another (normal) direct sum.

$$T(g) = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}$$

If $D(g) = 0$, then $U$ is an irreducible $G$-module.

$$U \text{ is irreducible } \iff (U, \text{inv}) \iff (C(g) = 0, \forall g \in G)$$

Irreps are completely reducible.
$\dim V < \infty$ \quad $K = \mathbb{R}$ or $\mathbb{C}$

Week 1 - 6 \quad $|G| < \infty$ \quad LA, a bit of Group Theory

Week 7 - MT

Week 8 - 13 \quad Lie groups and algebras \quad LA, Multivariable Calculus

Week 14 \quad Revision before the final

Week 15 \quad Final exam