Assume that $T'$ is not irreducible, i.e., $W = \text{null } T$ is non-trivial. Consider $\text{null } W = \{v \in V : T'(v) = 0, T(v) \neq 0\}$, and let $W$ be the complement of $\text{null } W$.

Let $A = \text{End}(V)$, $A^t = \text{End}(V^*)$.

The matrix of $A^t$ in the dual basis is the transpose of the matrix of $A$.

Theorems and definitions:

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta 
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$

Theorem 1 (Irreducibility):

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$

Theorem 2 (Decomposition):

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$

Theorem 3 (Isomorphism):

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$

Proof of Theorem 3 (Irreducibility):

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$

Proof of Theorem 4 (Decomposition):

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$

Proof of Theorem 5 (Isomorphism):

- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
- $T(\alpha, \beta) = T'(\alpha, \beta)$ for all $\alpha, \beta$
1. Definition of a representation

**Definition 1.1.** Assume that \( T : \mathbb{C} \rightarrow \mathbb{C}(T) \) is a representation over a field \( T \). Define the **character** \( \chi \) of the representation \( T \) by

\[
\chi_T = \text{tr}(T(x)), \quad x \in G.
\]

**Proposition 1.2.** If \( \chi_1 = \chi_2 \), then \( T_1 \) and \( T_2 \) have the same character.

2. Decomposition of a representation

**Theorem 2.1.** Denote by \( \Theta(x) \) the field \( \mathbb{C}(x) \) if \( x \in G \) or the field \( \mathbb{C} \) if \( x \notin G \). Assume that \( T : \mathbb{C} \rightarrow \mathbb{C}(T) \) is a representation of the group \( G \) over the field \( T \). If \( T \) is not equal to \( \mathbb{C} \) or \( \mathbb{C}(x) \) for any \( x \) in \( G \), then the representation \( T \) is completely reducible.

**Exercise 2.2.** Assume \( T : \mathbb{C} \rightarrow \mathbb{C}(T) \) is a representation of the group \( G \) over the field \( T \). Show that \( T \) is a direct sum of irreducible representations.

We claim that every real or complex representation \( T \) of a finite group \( G \) is isomorphic to a sum of one-dimensional representations with some multiplicities.

\[
T = \bigoplus_{i=1}^{n} \mathbb{C} \cdot e_i
\]

The multiplicities are uniquely determined by \( T \). This result is a bit technical; see [Valence, Theorem 1 of Section 2.1].

---

*Anon, October 30, 2020*
Assume \( \langle x, y \rangle \) is a group. \( \mathcal{G} = \langle x, y \rangle \) is the minimal normal subgroup that the quotient group of \( G \) by \( \mathcal{G} \) is abelian.