\[ \chi_{\phi_k}(s^m) = 0, \quad 0 \leq m < n - 1 \]

\[ \psi_{\phi_k}(a^m) = 2 \cos \frac{2\pi km}{h} \]

\[ ( \theta_k i8 i2) \iff ( \sin \frac{2\pi k}{h} = 0 ) \]

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**Math in Moscow**

**Homework Assignment 5**

(Due Date: October 16, 2020)

Recall that the dihedral group \( D_n \) is generated by a rotation \( a \) and a reflection \( s \) such that \( a^n = s^2 = e \), \( as = sa^{-1} \).

Let \( k \in \mathbb{Z} \). Define a \( 2 \)-dimensional complex representation \( \rho_k \) of the group \( D_n \) as

\[ \rho_k(a) = \begin{pmatrix} \cos \frac{2\pi k}{n} & \sin \frac{2\pi k}{n} \\ -\sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{pmatrix}, \quad \rho_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

**Problem 5.1.**

a) Check that the representation \( \rho_k \) is well-defined for all integer \( k \) and find its character.

b) When is the representation \( \rho_k \) irreducible?

**Problem 5.2.**

Construct the complex character table of the group \( D_n \).

**Problem 5.3.**

Construct the complex character table of the group \( S_1 \).

**Problem 5.4.**

Decompose every tensor product of two irreducible complex representations of the group \( S_1 \) into its irreducible representations.

**Problem 5.5.**

Consider a set \( F \) of complex functions on the set \( M \) of the faces of the cube. It is a two-dimensional complex space, and \( F \) is an irreducible representation of the dihedral group \( D_4 \). Find its character table.

---
If \( p=0 \) then
the conjugacy classes are
\[ \{e\}, \{a, a^5\}, \{a^2, a^4\}, \{a^3\} \]
If \( p=1 \)
\[ \{s, sa^2, sa^4\}, \{s, sa^3, sa^5\} \]
$\kappa_{P_0}(a) = 2$

$\kappa_{P_3}(a) = -2$

$\kappa_{P_4}(4m) = 2 \cos \frac{8\pi m}{6} = 2 \cos \left( \frac{12\pi m}{6} - \frac{4\pi m}{6} \right) = 2 \cos \frac{4\pi m}{6} = \kappa_{P_2}(a m)$

$P_0$, not i\$

$P_1$, in

$P_2$, in

$P_3$, i\$

$P_4$, not i\$

$P_5$, in

$P_6$, i\$

$P_7$, i\$

$P_1, P_2$ are i\$

$\kappa_{P_1}(a) \neq \kappa_{P_2}(a)$, therefore

$P_1 \not\sim P_2$
\[ C \cong (\mathbb{T} \circ \text{sgn}) \circ (\mathbb{T} \circ \text{sgn}) \cong \mathbb{T} \circ \mathbb{T} \]

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\[ \begin{array}{c|ccc|ccc|ccc|ccc}
    & S_4 & e & (12) & (123) & (1234) & (12)(34) \\
\hline
    IA & 1 & 1 & 1 & 1 & 1 & 1 \\
    S_4h & 1 & -1 & 1 & -1 & 1 & -1 \\
    C & 3 & -1 & 0 & 1 & -1 & -1 \\
    3 & 3 & 0 & -1 & 0 & 2 & -1 \\
    s_\Delta & 3 & 1 & 0 & 1 & 3 & 0 \\
\end{array} \]

\[ \text{sgn} \circ \mathbb{T} = \begin{pmatrix} 3 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 2 \end{pmatrix} \]

\[ \Delta = \Delta + \text{Id} \]

\[ 3 \Delta \cong \Delta + \text{Id} \]

\[ p : S_4 \to S_3 \]

\[ \ker p = \langle (12)(34), (13)(24), (14)(23) \rangle \]

\[ S_4 \cong S_3 \]

\[ \text{sgn} \circ \mathbb{T} \]
1) \[ \# \{(\xi, \eta) \in \mathbb{C}^2, \xi \neq \eta \} = 16 \]

2) \[ 2 \sum (\lambda_i T^2) = 16 \]

\[ T(g) \text{ is diagonalizable, assume } \lambda_1, \lambda_2, \ldots, \lambda_n \text{ are the diagonal entries in that diagonal form} \]

\[ T_z(T(g)) = \left( \sum_{i=1}^{n} \lambda_i \right)^2 \]

\[ T_z(T^2(g)) = \left( \sum_{i=1}^{n} \lambda_i \right)^2 \]

\[ \lambda_i \lambda_j \]

\[ T_z(S^2 T(g)) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \sum_{i=1}^{n} \lambda_i^2 + \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \]

\[ \lambda_i \lambda_j \]

\[ x_{S^2 T(g)} = \frac{(k_T(g))^2 + x_T(g^2)}{2} \]

\[ x_{H^2 T(g)} = \frac{(k_T(g))^2 - k_T(g^2)}{2} \]

\[ (k_T(g))^2 = \left( \sum_{i=1}^{n} \lambda_i \right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \]
If $\dim T = 1$, then $S^2 T \cong T$, $N^2 T \cong \mathbb{C}$.