Problem 4.1. Consider the projection operator $P \in \text{End}(\mathbb{R}^2 \otimes \mathbb{R}^2)$ onto the subspace $S^2 \mathbb{R}^2$ corresponding to the decomposition

$$\mathbb{R}^2 \otimes \mathbb{R}^2 = S^2 \mathbb{R}^2 \oplus \Lambda^2 \mathbb{R}^2.$$ Find the matrix of $P$ in the standard basis.

Problem 4.2. Let $T$ be a representation of a group $G$. Show that

$$\chi_{S^2 T} + \chi_{\Lambda^2 T} = (\chi_T)^2.$$ 

Problem 4.3. Find the character of the representation of the group $S_4$ in $\mathbb{R}^3$ determined by the motions of the cube.

Problem 4.4. a) Is the representation of the group $S_4$ in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube? 

b) Is the representation of the group $A_4$ in $\mathbb{R}^3$ determined by the motions of the regular tetrahedron isomorphic to the representation of the same group determined by the motions of the cube?

Problem 4.5*. Suppose $T$ is an irreducible representation of the group $G$ over $\mathbb{R}$ and $\dim T$ is odd. Show that the complexification $T^C$ is irreducible as the representation of the group $G$ over $\mathbb{C}$.

Based on: Fulton, Harris, Sections 1.2, 1.3, Vinberg, Sections 1.2-1.4, A1, James, Liebeck, Sections 10, 12, 13.

Date: October 2, 2020.