LECTURE 1. THE NOTION OF A REPRESENTATION.

1. Preliminary Definitions

To give a definition of a representation we need to introduce our actors first. We start with the notion of a group.

Definition 1.1. *A set G with maps $* : G \times G \to G$ (notation $*(a,b) = ab$) and $inv : G \to G$ (notation $inv(a) = a^{-1}$) is called a group if the following properties hold*

(1) $(ab)c = a(bc), \quad \forall a, b, c \in G$;
(2) $\exists e \in G$ *such that* $ae = ea = a, \quad \forall a \in G$;
(3) $\forall a \in G, \exists b \in G$ *such that* $ab = ba = e$.

Question 1.2. *Does the definition of a group imply that an identity element is unique ?*

Definition 1.3. *Assume G and H are groups. The map $f : G \to H$ is called a group homomorphism if $f(g_1g_2) = f(g_1)f(g_2)$, $\forall g_1, g_2 \in G$.*

Question 1.4. *Assume $f : G \to H$ is a homomorphism. Is it always true that $f(e_G) = e_H$ and $f(g^{-1}) = (f(g))^{-1}$, $\forall f \in G$?*

Definition 1.5. *A group G is called* abelian *if $ab = ba, \quad \forall a, b \in G$.*

Question 1.6. *Recall a definition of a field using the notion of an abelian group.*

Question 1.7. *Assume $\mathbb{F}$ is a field. Recall the definition of a vector space over the field $\mathbb{F}$?*

Question 1.8. *Assume $\mathbb{F}$ is a field, V is a vector space over the field F. Show that*

$$0_{\mathbb{F}} \cdot v = 0_V, \quad \forall v \in V, \tag{1.1}$$

$0_{\mathbb{F}}, 0_V$ are the identity elements of the corresponding abelian groups.

Question 1.9. *Assume V and U are vector spaces over a field $\mathbb{F}$. When do we call a map $A : V \to U$ linear ?*

Question 1.10. *Assume V is a vector spaces over a field $\mathbb{F}$. Do the invertible linear maps from V to itself form a group with respect to the operation of maps compostition ?*

Definition 1.11. *Assume V is a vector spaces over a field $\mathbb{F}$. The set of invertible linear maps from V to itself is called general linear group and is denoted by $GL(V)$.*

Question 1.12. *Is $GL(V)$ a vector space with respect to naturally defined addition and multiplication by scalars ?*

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2. Representations and their invariant subspaces

Definition 2.1. *Assume G is a group, $\mathbb{F}$ is a field, V is a vector space over $\mathbb{F}$. Then a homomorphism*

$$T : G \to GL(V) \tag{2.1}$$

is called a representation of the group G (in the vector space V) (over the field $\mathbb{F}$). The dimension of the representation T is, by definition, the dimension of the vector space V.

Question 2.2. *For any given group G and vector space V does there always exist a representation of G in V ?*

Problem 2.3. *Find all 1-dimensional representations of the cyclic group $\mathbb{Z}_n$*
a) over $\mathbb{R}$,   b) over $\mathbb{C}$.

Problem 2.4. *When the map $T : \mathbb{Z} \to \mathrm{Mat}(2 \times 2, \mathbb{R})$ given by the formula*

a) $T(n) = \begin{pmatrix} an & 0 \\ 0 & bn \end{pmatrix}$ *b)* $T(n) = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ *c)* $T(n) = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$

d) $T(n) = \begin{pmatrix} 1 & a^n \\ 0 & 1 \end{pmatrix}$ *e)* $T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}$ $a,\, b,\, \alpha \in \mathbb{R}$.

is a representation of the group $\mathbb{Z}$ over the field $\mathbb{R}$?

Now let us define a sum of two representations. First we recall a construction from linear algebra.

Definition 2.5. *Assume V and U are vector spaces over $\mathbb{F}$. The Cartesian product $V \times U$ with respect to operations of component-wise addition and multiplication by scalars is called the (external) direct sum of V and U and is denoted by $V \oplus U$.*

Now we can define the sum of two representations of the same group over the same field.

Definition 2.6. *Assume $T : G \to GL(V)$ and $S : G \to GL(U)$ are representations over $\mathbb{F}$. Then define $T \oplus S : G \to GL(V \oplus U)$ via*

$$(T \oplus S)(g)(v, u) := (T(g)v, S(g)u), \quad \forall g \in G, v \in V, u \in U. \tag{2.2}$$

Assume that we suspect that a representation is a sum of some representations of smaller dimensions. How should we proceed to find those smaller blocks ? It would be natural to give the following definition.

Definition 2.7. *Assume $T : G \to GL(V)$ is a representation. A subspace $U \subset V$ is called invariant wrt T if*

$$T(g)u \in U, \quad \forall g \in G, u \in U. \tag{2.3}$$

If $U \neq \{0\}$, $U \neq V$ then U is called non-trivial.

Note that we can construct the representation $S : G \to GL(U)$ setting $S(g)$ to be the restriction of the automorphism $T(g)$ to the subspace U. S is called a subrepresentation of the representation T.

Problem 2.8. *Describe the non-trivial invariant subspaces for the representations from the Exercise 2.4.*

Problem 2.9. *Consider the representation* $T : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2)$ *given by the formula*

$$T(k) = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin\frac{2\pi k}{n}) \\ -\sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Find all its non-trivial invariant subspaces.

Question 2.10. *Assume* $T : G \to GL(V)$ *is a representation,* $U \subset V$ *is an invariant subspace,* $u_1, \ldots, u_k$ *is a basis of* U*,* $u_1, \ldots, u_k, w_1, \ldots, w_m$ *is a basis of* V*. We can express* $T(g)$ *in a block-diagonal form:*

$$T(g) = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}. \tag{2.4}$$

How can we reformulate the fact that U *is an invariant subspace in terms of* $A(g), B(g), C(g), D(g)$ *?*

Definition 2.11. *Assume* $T : G \to GL(V)$ *is a representation.* T *is called irreducible if it has no non-trivial invariant subspaces.*

Definition 2.12. *Assume* $T : G \to GL(V)$ *is a representation.* T *is called completely reducible if for any invariant subspace* U *there exist another invariant subspace* W *such that* $V = U \oplus W$*. Here the notation* $\oplus$ *stands for another (internal) direct sum.*

Question 2.13. *Is it true that any irreducible representation is completely reducible ?*

Question 2.14. *Does there exist a representation that is not completely reducible ?*

Problem 2.15. *Define a representation* T *of the symmetric (permutation) group* S_n *in* $\mathbb{C}^n$ *by*

$$T(\sigma)((x_1, \ldots, x_n)) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

for $\sigma \in S_n$*. Find all non-trivial invariant subspaces of the representation* T*.*

Chapters from the textbooks relevant for the lecture:

- Fulton, Harris, Section 1.1;
- James, Liebeck, Chapters 1 - 5;
- Vinberg, Sections 1.1 - 1.3.