LECTURE 2. SCHUR'S LEMMA AND COMPLETE REDUCIBILITY

1. Irreducible and completely reducible representations

Question 1.1. *Assume $T : G \to GL(V)$ is a representation, $U \subset V$ is an invariant subspace, $u_1, \dots, u_k$ is a basis of U, $u_1, \dots, u_k, w_1, \dots, w_m$ is a basis of V. We can express $T(g)$ in a block-diagonal form:*

$$T(g) = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}. \tag{1.1}$$

How can we reformulate the fact that U is an invariant subspace in terms of $A(g), B(g), C(g), D(g)$?

Definition 1.2. *Assume $T : G \to GL(V)$ is a representation. T is called irreducible if it has no non-trivial invariant subspaces.*

Definition 1.3. *Assume $T : G \to GL(V)$ is a representation. T is called completely reducible if for any invariant subspace U there exist another invariant subspace W such that $V = U \oplus W$. Here the notation $\oplus$ stands for (internal) direct sum.*

Question 1.4. *Is it true that any irreducible representation is completely reducible ?*

Question 1.5. *Does there exist a representation that is not completely reducible ?*

2. Morphisms of representations and Schur's Lemma

Question 2.1. *Assume that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations over the same field $\mathbb{F}$. When would it be natural to say that T and S are isomorphic ?*

Definition 2.2. *Assume that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations. $A \in \mathrm{Hom}(V, U)$ is called a (homo-)morphism of representations T and S if*

$$AT(g) = S(g)A, \quad \forall g \in G. \tag{2.1}$$

We are going to denote the set of morhisms of representations T and S by $\mathrm{Mor}(T, S)$ (by $\mathrm{End}(T)$ if $S = T$).

Question 2.3. *How to reformulate the definition of a morphism of representations if we consider those representations as G-modules ?*

Question 2.4. *Is $\mathrm{End}(T)$ a subspace of $\mathrm{End}(V)$?*

Exercise 2.5. *Describe all morphisms between two one-dimensional complex representations of the group $\mathbb{Z}_n$.*

Problem 2.6. *Assume that $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^2)$ is defined as*

$$T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix},$$

where $\alpha \neq k\pi$, $k \in \mathbb{Z}$. Describe $\mathrm{End}(T)$.

Date: September 15, 2022.

Problem 2.7. *Assume that* $T:\mathbb{Z}\to \mathrm{GL}(\mathbb{C}^2)$ *and* $S:\mathbb{Z}\to \mathrm{GL}(\mathbb{C}^2)$ *are defined as*

$$T(n)=\begin{pmatrix}\cos(n\alpha) & \sin(n\alpha)\\ -\sin(n\alpha) & \cos(n\alpha)\end{pmatrix},\quad S(n)=\begin{pmatrix}\exp(in\alpha) & 0\\ 0) & \exp(-in\alpha)\end{pmatrix}.$$

Describe $\mathrm{Mor}(T,S)$*.*

Problem 2.8. *Assume that* $T:\mathbb{Z}_4\to \mathrm{GL}(\mathbb{C}^3)$ *and* $S:\mathbb{Z}_4\to \mathrm{GL}(\mathbb{C}^3)$ *are defined as*

$$T(n)=\begin{pmatrix}1 & 0 & 0\\ 0 & (-1)^n & 0\\ 0 & 0 & i^n\end{pmatrix},\quad S(n)=\begin{pmatrix}(-1)^n & 0 & 0\\ 0 & i^n & 0\\ 0 & 0 & i^n\end{pmatrix}.$$

Describe $\mathrm{Mor}(T,S)$ *and* $\mathrm{End}(S)$*.*

Definition 2.9. *Assume* $T:G\to \mathrm{GL}(V)$ *and* $S:G\to \mathrm{GL}(U)$ *are representations. They are called isomorphic if* $\exists A\in \mathrm{Iso}(V,U)$ *such that* $A\in \mathrm{Mor}(T,S)$ *(*A *is called an isomorphism of representations in that case). The set of isomorhisms of representations* T *and* S *is denoted by* $\mathrm{Iso}(T,S)$*. The set of automorhisms of the representation* T *is denoted by* $\mathrm{Aut}(T)$*.*

Lemma 2.10. *Assume* $T:G\to \mathrm{GL}(V)$ *and* $S:G\to \mathrm{GL}(U)$ *are representations,* $A\in \mathrm{Mor}(T,S)$*. Then*

a) $\ker A=\{v\in V\,|\,Av=0\}\subset V$ *is an invariant subspace wrt* T*;*
b) $\mathrm{im}A=\{Av\,|\,v\in V\}\subset U$ *is an invariant subspace wrt* S*.*

Exercise 2.11. *Prove this lemma.*

Proposition 2.12. *Assume* $T:G\to \mathrm{GL}(V)$ *and* $S:G\to \mathrm{GL}(U)$ *are irreducible representations,* $A\in \mathrm{Mor}(T,S)$*. Then either* $A=0$ *or* $A\in \mathrm{Iso}(T,S)$*.*

Exercise 2.13. *Prove this proposition.*

Proposition 2.14. *(Schur's Lemma) Assume* $T:G\to \mathrm{GL}(V)$ *is an irreducible complex representations,* $A\in \mathrm{End}(T)$*. Then* $\exists\lambda\in\mathbb{C}$ *such that* $A=\lambda\mathrm{Id}$*.*

Exercise 2.15. *Prove this proposition.*

Problem 2.16. *Is it true that every irreducible complex representation of an abelian group is 1-dimensional ?*

Question 2.17. *Is the claim of Schur's Lemma still true over* $\mathbb{R}$ *?*

Definition 2.18. *A vector space* V *over a field* $\mathbb{F}$ *with an operation* $\cdot : V\times V\to V$ *is called an algebra if the following properties hold:*

(1) $a(b+c)=ab+ac,\quad \forall a,b,c\in V;$
(2) $(b+c)a=ba+bc,\quad \forall a,b,c\in V;$
(3) $\lambda(ab)=(\lambda a)b=a(\lambda b),\quad \forall a,b\in V,\ \forall\lambda\in\mathbb{F}.$

An algebra V *is called associative if* $a(bc)=(ab)c$*,* $\forall a,b,c\in V$*. An associative algebra* V *is called a division algebra if* $(V\setminus\{0\},\cdot)$ *is a group.*

Question 2.19. *Assume* $T:G\to \mathrm{GL}(V)$ *is an irreducible representation over* $\mathbb{F}$*. Does* $\mathrm{End}(T)$ *have to be a division algebra over* $\mathbb{F}$ *?*

Question 2.20. *What are possible division algebras over* $\mathbb{C}$ *?*

Question 2.21. *What are possible division algebras over $\mathbb{R}$?*

Problem 2.22. *Construct an irreducible real representation of a finite group such that its algebra of endomorphisms is isomorphic to the division algebra of quaternions. You are free to pick a finite group you prefer.*

3. Complete reducibility of representations

Problem 3.1. *Show that there exists a unique representation $T : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^2)$ such that*

$$T(1) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Express T as a direct sum of irreducible subrepresentations.

Definition 3.2. *Let $\mathbb{F}$ be a field. Then we denote by $\mathrm{char}\mathbb{F}$ the minimal natural number n such that*

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0. \tag{3.1}$$

If there is no such natural number we put $\mathrm{char}\mathbb{F} := 0$.

Proposition 3.3. *(Maschke) Assume that a group G is finite, $\mathbb{F}$ is a field, V is a vector space over the field $\mathbb{F}$, $T : G \to \mathrm{GL}(V)$ is a representation, and $\mathrm{char}\mathbb{F}$ does not divide $|G|$. Then the representation T is completely reducible.*

Proof. Assume $U \subset V$ is an invariant subspace. Denote by $W \subset V$ a subspace of V such that $U \oplus W = V$. Define $P \in \mathrm{End}(V)$ by setting

$$P(u) = u, \quad \forall u \in U, \quad P(w) = 0, \quad \forall w \in W. \tag{3.2}$$

Now we will average P over the group G. Put

$$P_0 = \frac{1}{|G|} \sum_{g \in G} T(g) P T(g^{-1}). \tag{3.3}$$

Then $P_0 \in \mathrm{End}(T)$. Thus, $\ker(P_0)$ is invariant wrt T. One can show that

$$U \oplus \ker P_0 = V. \tag{3.4}$$

□

Exercise 3.4. *Show the last claim in the proof.*

Question 3.5. *Assume $T : G \to \mathrm{GL}(V)$ is a completely reducible representation and V is expressed as a sum of minimal invariant subspaces:*

$$V = \bigoplus_{i=1}^{m} V_i \tag{3.5}$$

Is such expression unique up to permutation of summands ?

Chapters from the textbooks relevant for the lecture:

- Fulton, Harris, Sections 1.1, 1.2;
- James, Liebeck, Chapters 7 - 9;
- Vinberg, Sections 1.1 - 1.4.