LECTURE 3. DUAL REPRESENTATION AND TENSOR PRODUCT OF REPRESENTATIONS

1. Complete reducibility of representations

Definition 1.1. *Let $\mathbb{F}$ be a field. Then we denote by* $\mathrm{char}\mathbb{F}$ *the minimal natural number n such that*

$$\underbrace{1+1+\cdots+1}_{n\ times}=0. \tag{1.1}$$

If there is no such natural number we put $\mathrm{char}\mathbb{F}:=0$.

Proposition 1.2. *(Maschke) Assume that a group G is finite, $\mathbb{F}$ is a field, V is a vector space over the field $\mathbb{F}$, $T: G \to \mathrm{GL}(V)$ is a representation, and* $\mathrm{char}\mathbb{F}$ *does not divide $|G|$. Then the representation T is completely reducible.*

Proof. Assume $U \subset V$ is an invariant subspace. Denote by $W \subset V$ a subspace of V such that $U \oplus W = V$. Define $P \in \mathrm{End}(V)$ by setting

$$P(u)=u, \quad \forall u \in U, \quad P(w)=0, \quad \forall w \in W. \tag{1.2}$$

Now we will average P over the group G. Put

$$P_0=\frac{1}{|G|}\sum_{g\in G}T(g)PT(g^{-1}). \tag{1.3}$$

Then $P_0 \in \mathrm{End}(T)$. Thus, $\ker(P_0)$ is invariant wrt T. One can show that

$$U \oplus \ker P_0 = V. \tag{1.4}$$

□

Exercise 1.3. *Show the last claim in the proof.*

Problem 1.4. *Assume that the group G is finite and $T: G \to \mathrm{GL}(\mathbb{C}^2)$ is a representation. Suppose $\exists g,h \in G$ such that $T(g)T(h) \neq T(h)T(g)$. Can the representation T be not irreducible ?*

Question 1.5. *Assume $T: G \to \mathrm{GL}(V)$ is a completely reducible representation and V is expressed as a sum of minimal invariant subspaces:*

$$V=\bigoplus_{i=1}^{m} V_i \tag{1.5}$$

Is such expression unique up to permutation of summands ?

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2. Dual space and dual representation

First let us recall the algebraic construction.

Definition 2.1. *Let V be a vector space over a field $\mathbb{F}$. The vector space* $\mathrm{Hom}(V, \mathbb{F})$ *is called the dual space (to the vector space V) and is denoted by V^*.*

Definition 2.2. *Assume that V is a finite-dimensional vector space over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V. Define $f^i \in V^*$ via*

$$f^i(v_j) = \delta^i_j, \quad 1 \le i, j \le n \tag{2.1}$$

The set $\{f^1, f^2, \ldots, f^n\}$ is called the dual basis (to the basis $\{e_1, \ldots, e_n\}$).

Exercise 2.3. *Show that $\{f^1, f^2, \ldots, f^n\}$ is a basis of V^**

Proposition 2.4. *Assume that V is a finite-dimensional vector space over a field $\mathbb{F}$. Then there exists an isomorphism of V and $(V^*)^*$ such that it does not depend on the choice of basis in V.*

Exercise 2.5. *Prove this proposition.*

Question 2.6. *Can we make a similar statement about an isomorphism of V and V^* ?*

Definition 2.7. *Assume that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in Hom(V, U)$. Define $A^* \in \mathrm{Hom}(U^*, V^*)$ by*

$$(A^*(f))(v) = f(Av), \quad \forall v \in V, \forall f \in U^*. \tag{2.2}$$

Exercise 2.8. *Show that for any $A \in \mathrm{Hom}(V, U)$ the map $A^* \in \mathrm{Hom}(U^*, V^*)$ exists and it is unique.*

Problem 2.9. *Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V, $\{f^1, \ldots, f^n\}$ is its dual basis, $\{u_1, \ldots, u_m\}$ is a basis of U, $\{g^1, \ldots, g^m\}$ is its dual basis, $A \in Hom(V, U)$ and its matrix in the bases $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_m\}$ is*

$$(a^j_i)_{1 \le i \le n, 1 \le j \le m} \tag{2.3}$$

Find the matrix of A^* in the bases $\{g^1, \ldots, g^m\}$, $\{f^1, \ldots, f^n\}$

Definition 2.10. *Assume $T : G \to \mathrm{GL}(V)$ is a representation over a field F. Define its dual representation $T^* : G \to \mathrm{GL}(V^*)$ by*

$$T^*(g) = (T(g^{-1}))^*. \tag{2.4}$$

Problem 2.11. *Show that T^* is indeed a representation of the group G.*

Problem 2.12. *Define $T : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^3)$ as*

$$T(n) = \begin{pmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that T is a representation. Find the matrix of $T^*(n)$ in the dual basis.

Question 2.13. *Assume $T : G \to \mathrm{GL}(V)$ is a representation. Is it always true that the representations T and $(T^*)^*$ are isomorphic ?*

Problem 2.14. *Let T be an irreducible representation of a group G. Is it possible that the representation T^* is not irreducible ?*

3. Tensor products of vector spaces, endomorphisms and representations

Definition 3.1. *Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$. Denote by $Free(V, U)$ the set of finite linear combinations $\sum_{i=1}^{n} a_i(x_i, y_i)$, where $n \in \mathbb{Z}_+$, $x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \leq i \leq n$. It has a natural structure of a vector space. Then define $Ident(V, U) \subset Free(V, U)$ as*

$$Ident(V, U) = \mathrm{span}\langle \{(ax + bz, y) - a(x, y) - b(z, y) \,|\, x, z \in V, y \in U, a, b \in \mathbb{F}\} \cup \{(x, ay + bt) - a(x, y) - b(x, t) \,|\, x \in V, y, t \in U, a, b \in \mathbb{F}\}\rangle. \quad (3.1)$$

The tensor product $V \otimes U$ is, by definition, the quotient space $Free(V, U)$ by $Ident(V, U)$.

Proposition 3.2. *Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V, $\{u_1, \ldots, u_m\}$ is a basis of U. Then*

$$\{v_i \otimes u_j \,|\, 1 \leq i \leq n, 1 \leq j \leq m\} \quad (3.2)$$

is a basis of $V \otimes U$.

Exercise 3.3. *Prove this proposition, it may be a bit challenging to show the linear independence.*

Proposition 3.4. *Assume that V is a finite-dimensional vector space. Then there exists an isomorphism between the vector spaces $V \otimes V^*$ and $\mathrm{End}(V)$ such that it does not depend on the choice of the basis in V*

Exercise 3.5. *Prove this proposition.*

Definition 3.6. *Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(U)$. Define $A \otimes B \in \mathrm{End}(V \otimes U)$ by the formula*

$$A \otimes B(v \otimes u) := (Av) \otimes (Bu) \quad (3.3)$$

the tensors of the form $v \otimes u$ and then extend by linearity.

Exercise 3.7. *Show that $A \otimes B$ is well-defined.*

Exercise 3.8. *Assume that V, U are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \ldots, v_n\}$ is a basis of V, $\{u_1, \ldots, u_m\}$ is a basis of U, $A \in \mathrm{End}(V)$, its matrix in the basis $\{v_1, \ldots, v_n\}$ is*

$$(a_i^j)_{1 \leq i \leq n, 1 \leq j \leq n}, \quad (3.4)$$

$B \in \mathrm{End}(U)$, its matrix in the basis and $\{u_1, \ldots, u_m\}$ is

$$(b_i^j)_{1 \leq i \leq m, 1 \leq j \leq m}. \quad (3.5)$$

Find the matrix of $A \otimes B$ in the basis

$$\{v_1 \otimes u_1, v_1 \otimes u_2, \ldots, v_1 \otimes u_m, v_2 \otimes u_1, \ldots, v_n \otimes u_m\}. \quad (3.6)$$

Question 3.9. *Is it always true that*

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)? \quad (3.7)$$

Definition 3.10. *Assume $T : G \to \mathrm{GL}(V)$, $S : G \to \mathrm{GL}(U)$ are representations over a field $\mathbb{F}$. Then $T \otimes S : G \to \mathrm{GL}(V \otimes U)$ is defined by*

$$(T \otimes S)(g) := (T(g)) \otimes (S(g)). \tag{3.8}$$

Exercise 3.11. *Let $T_1, \dots, T_n$ be the set of all pairwise non-isomorphic one-dimensional complex representations of G. Show that this set form a group, where the tensor product plays a role of the multiplication and the inverse element is the dual representation. Describe that group for $G = \mathbb{Z}_n$, S_n.*

Problem 3.12. *Let T be a representation of a finite group G. Is it true that the representation $T \otimes T^*$ of the group G always has a one-dimensional invariant subspace ?*

4. Symmetric and antisymmetric tensors

Definition 4.1. *Let V be a vector space over $\mathbb{F}$. Define the subspaces $S^2(V)$ and $\Lambda^2(V)$ of $V \otimes V$ as*

$$S^2(V) := \mathrm{span}\langle x \otimes y + y \otimes x \,|\, x, y \in V\rangle, \quad \Lambda^2(V) := \mathrm{span}\langle x \otimes y - y \otimes x \,|\, x, y \in V\rangle. \tag{4.1}$$

Exercise 4.2. *Show that for any finite-dimensional vector space V we have that*

$$S^2(V) \oplus \Lambda^2(V) = V \otimes V. \tag{4.2}$$

Problem 4.3. *Consider the projection operator $P \in \mathrm{End}(\mathbb{R}^2 \otimes \mathbb{R}^2)$ onto the subspace $S^2\mathbb{R}^2$ corresponding to the decomposition*

$$\mathbb{R}^2 \otimes \mathbb{R}^2 = S^2\mathbb{R}^2 \oplus \Lambda^2\mathbb{R}^2.$$

Find the matrix of P in the standard basis.

Definition 4.4. *Assume that V is a vector space, $A \in \mathrm{End}(V)$ over $\mathbb{F}$. Define the endomorhisms $S^2(A) \in \mathrm{End}(S^2(V)), \Lambda^2(A) \in \mathrm{End}(\Lambda^2(V))$ as the restrictions of $A \otimes A$ to the subspaces $S^2(V)$ and $\Lambda^2(V)$ respectively.*

Definition 4.5. *Assume that $T : G \to \mathrm{GL}(V)$ is a representation. Define representations $S^2T : G \to \mathrm{GL}(S^2(V)), \Lambda^2T : G \to \mathrm{GL}(\Lambda^2(V)$ by*

$$(S^2T)(g) := S^2(T(g)), \quad (\Lambda^2T)(g) := \Lambda^2(T(g)). \tag{4.3}$$

Chapters from the textbooks relevant for the lecture:

- S.Roman, Advanced Linear Algebra, Sections 3,14.