LINEAR ALGEBRA PROBLEM SET, DUE SEPTEMBER 14

1. Vector spaces

Problem 1.1. *Suppose that U_1, U_2 and U_3 are subspaces of a vector space V. Is it true that $V = U_1 \oplus U_2 \oplus U_3$ if and only if the following conditions hold:*

(1) $V = U_1 + U_2 + U_3$;
(2) $U_1 \cap U_2 = \{0\}$;
(3) $U_1 \cap U_3 = \{0\}$;
(4) $U_2 \cap U_3 = \{0\}$?

Problem 1.2. *Let U be a subspace of a finite-dimensional vector space V. Show that the quotient space V/U is also finite-dimensional and express its dimension using* $\dim V$ *and* $\dim U$.

Problem 1.3. *Denote by* $\mathrm{Mat}(n \times n, \mathbb{R})$ *the vector space of real-valued $n \times n$ matrices, and by U the subspace of symmetric matrices.*

a) Find a basis of U.
b) Find a subspace W such that

$$\mathrm{Mat}(n \times n, \mathbb{R}) = U \oplus W.$$

Problem 1.4. *Do there exist vector spaces V, U and linear maps $A \in \mathrm{Hom}(V, U)$, $B \in \mathrm{Hom}(U, V)$ such that $AB = \mathrm{Id}$ and $A \notin \mathrm{Iso}(V, U)$?*

Problem 1.5. *What is "the simplest possible matrix" we can obtain for an arbitrary $A \in \mathrm{Hom}(V, U)$ by picking suitable bases in the vector spaces V and U?*

Problem 1.6. *Consider ab arbitrary $m \times n$ matrix A. Is it always true that the dimensions of the linear spans of its rows and of its columns are equal?*

Problem 1.7. *Let V be a finite-dimensional space over a field $\mathbb{F}$. Assume that $A \in End(V)$ such that*

$$AB = BA, \quad \forall B \in \mathrm{End}(V).$$

Does there always exist $\lambda \in \mathbb{F}$ such that $A = \lambda \mathrm{Id}$?

Problem 1.8. *Suppose that P is a linear operator in a vector space V. Show that $P^2 = P$ if and only if the following two conditions hold:*

(1) $V = \ker P \oplus \mathrm{im} P$;
(2) $Px = x, \quad \forall x \in \mathrm{im} P$.

P is called a projection operator.

Problem 1.9. *Suppose that* $\dim V < \infty$ *and $A, B \in \mathrm{End}(V)$. Then*

$$\det(AB) = \det(A) \cdot \det(B).$$

Problem 1.10. *Suppose that v is a vector space over a field $\mathbb{F}$, $\lambda \in \mathbb{F}$, $A, B \in \mathrm{End}(V)$, $AB = BA$, and*

$$U = \{x \in V \mid Ax = \lambda x\}.$$

Does U have to be B-invariant?

Problem 1.11. *Assume that $v_1, \ldots, v_k$ are eigenvalues of a linear operator A with distinct eigenvalues, then the set $\{v_1, \ldots, v_k\}$ is linearly independent.*

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Problem 1.12. *Show that*

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad \forall A, B \in \operatorname{End}(V).$$

Problem 1.13. *Let V be a finite-dimensional vector space. Suppose that $f \in (\operatorname{End}(V))^*$ does not depend on the choice of the basis, i.e.*

$$f(CAC^{-1}) = f(A), \quad \forall A \in \operatorname{End}(V), \forall C \in \operatorname{GL}(V).$$

Show that $\exists c \in \mathbb{F}$ such that $f(A) = c \cdot \operatorname{tr}(A)$, $\forall A \in \operatorname{End}(V)$.

Problem 1.14. *For which k and n does there exist $A \in \operatorname{End}(\mathbb{C}^n)$ such that $A^{k+1} = A^k$ and $A^k \neq A^{k-1}$?*

Problem 1.15. *Does there exist $A \in \operatorname{End}(\mathbb{R}^7)$ such that $A^2 + A + \operatorname{Id}$ is nilpotent?*