LINEAR ALGEBRA PROBLEM SET 2, DUE SEPTEMBER 15

2. Inner product spaces

We consider only finite-dimensional vector spaces.

Problem 2.1. *Is it true that every orthogonal set of nonzero vectors is linearly independent?*

Problem 2.2. *Suppose that* $\{v_1, \dots, v_k\}$ *is an orthogonal set of nonzero vectors in an inner product space* V*,* $w \notin \operatorname{span}\langle v_1, \dots, v_k\rangle$*. Then there exists a unique, up to multiplication by a scalar, vector* v_{k+1} *satisfying the following properties:*

a) The set $\{v_1, \dots, v_k, v_{k+1}\}$ *is an orthogonal set of nonzero vectors;*

b) $v_{k+1} \in \operatorname{span}\{v_1, \dots, v_k, w\}$*.*

Problem 2.3. *Suppose* V *is a finite-dimensional inner product space,* U *is its subspace. Then*

$$V = U \oplus U^{\perp}$$

and

$$(U^{\perp})^{\perp} = U.$$

Problem 2.4. *Suppose that* $\{v_1, \dots, v_n\}$ *is an orthonormal basis of an inner product space* V*. Then for every vector* $x \in V$*, the following decomposition holds*

$$x = \sum_{j=1}^{n} \langle x, v_j\rangle v_j.$$

Problem 2.5. *Define an inner product in the vector space* $\operatorname{Mat}(n \times n, \mathbb{C})$ *by*

$$\langle A, B\rangle := \operatorname{tr}((\overline{B})^t A).$$

Check that it is an inner product and find some orthonormal basis of $\operatorname{Mat}(n \times n, \mathbb{C})$*.*

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Problem 2.6. *Suppose that U and W are subspaces of a finite-dimensional inner product space V. Is it always true that*

$$(U+W)^{\perp} = U^{\perp} \cap W^{\perp} \quad ?$$

Problem 2.7. *Suppose that V is a finite-dimensional inner product space and $A \in \mathrm{End}(V)$. Then*

$$(\ker(A))^{\perp} = \mathrm{im}(A^*).$$

Problem 2.8. *For every unitary operator A in a finite-dimensional complex inner product space V there exists a basis α in which the matrix of A is diagonal:*

$$[A]_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \quad \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C},\ |\lambda_j| = 1, 1 \le j \le n.$$

Problem 2.9. *For every orthogonal operator A in a finite-dimensional real inner product space V there exists a basis α in which*

$$[A]_\alpha = \begin{pmatrix} R_{\psi_1} & 0 & \dots & 0 & 0 & 0 \\ 0 & R_{\psi_2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{\psi_r} & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathrm{Id}_p & 0 \\ 0 & 0 & \dots & 0 & 0 & -\mathrm{Id}_q \end{pmatrix}, \quad p, q, r \ge 0,\ \psi_1, \psi_2, \dots, \psi_r \in \mathbb{R},$$

and by Id_p we denote the identity $p \times p$ matrix. The sizes of zero matrices in the off-diagonal entries vary depending on the sizes of the corresponding blocks on the main diagonal.

Problem 2.10. *For every symmetric form in a finite-dimensional vector space over $\mathbb{R}$ there exists a basis in which its matrix has the form*

$$\begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ 0 & -\mathrm{Id}_l & 0 \\ 0 & 0 & 0_m \end{pmatrix}.$$

The numbers k, l, and m are uniquely determined by the bilinear form.