LECTURE 1. THE NOTION OF A REPRESENTATION.

1. Preliminary Definitions

To define a representation we need to introduce our actors first. We start with the notion of a group.

Definition 1.1. *A set G with the maps $* : G \times G \to G$ (notation $*(a, b) = ab$) and $inv : G \to G$ (notation $inv(a) = a^{-1}$) is called a group if the following properties hold*

(1) $(ab)c = a(bc), \quad \forall a, b, c \in G$;
(2) $\exists e \in G$ *such that* $ae = ea = a, \quad \forall a \in G$;
(3) $\forall a \in G, \exists b \in G$ *such that* $ab = ba = e$.

Question 1.2. *Does the definition of a group imply that an identity element is unique?*

Definition 1.3. *Suppose that G and H are groups. A map $f : G \to H$ is called a group homomorphism if $f(g_1g_2) = f(g_1)f(g_2)$, $\forall g_1, g_2 \in G$.*

Question 1.4. *Let $f : G \to H$ be a homomorphism. Is it always true that $f(e_G) = e_H$ and $f(g^{-1}) = (f(g))^{-1}$, $\forall g \in G$?*

Definition 1.5. *A group G is called abelian if $ab = ba, \quad \forall a, b \in G$.*

Question 1.6. *Recall a definition of a field using the notion of an abelian group.*

Question 1.7. *Let $\mathbb{F}$ be a field. Recall the definition of a vector space over the field $\mathbb{F}$.*

Question 1.8. *Suppose that $\mathbb{F}$ is a field and V is a vector space over the field F. Show that*

$$0_{\mathbb{F}} \cdot v = 0_V, \quad \forall v \in V, \tag{1.1}$$

$0_{\mathbb{F}}, 0_V$ are the identity elements of the corresponding abelian groups.

Question 1.9. *Assume that V and U are vector spaces over a field $\mathbb{F}$. When do we call a map $A : V \to U$ linear?*

Question 1.10. *Let V be a vector space over a field $\mathbb{F}$. Show that the set of invertible linear maps from V to itself is a group with respect to the operation of maps composition?*

Definition 1.11. *Let V be a vector space over a field $\mathbb{F}$. The set of invertible linear maps from V to itself is called a general linear group and is denoted by $GL(V)$.*

Question 1.12. *Is $GL(V)$ a vector space with respect to naturally defined addition and multiplication by scalars?*

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2. Representations and their invariant subspaces

Definition 2.1. *Suppose that G is a group, $\mathbb{F}$ is a field, and V is a vector space over $\mathbb{F}$. Then a homomorphism*

$$T : G \to GL(V) \tag{2.1}$$

is called a representation of the group G (in the vector space V) (over the field $\mathbb{F}$). The dimension of the representation T is, by definition, the dimension of the vector space V.

Question 2.2. *For any given group G and vector space V does there always exist a representation of G in V?*

Problem 2.3. *Find all 1-dimensional representations of the cyclic group $\mathbb{Z}_n$*
a) over $\mathbb{R}$, b) over $\mathbb{C}$.

Problem 2.4. *When the map $T : \mathbb{Z} \to \mathrm{Mat}(2 \times 2, \mathbb{R})$ given by the formula*

a) $T(n) = \begin{pmatrix} an & 0 \\ 0 & bn \end{pmatrix}$ *b)* $T(n) = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ *c)* $T(n) = \begin{pmatrix} 1 & an \\ 0 & 1 \end{pmatrix}$

d) $T(n) = \begin{pmatrix} 1 & a^n \\ 0 & 1 \end{pmatrix}$ *e)* $T(n) = \begin{pmatrix} \cos(n\alpha) & \sin(n\alpha) \\ -\sin(n\alpha) & \cos(n\alpha) \end{pmatrix}$ $a,\ b,\ \alpha \in \mathbb{R}$.

is a representation of the group $\mathbb{Z}$ over the field $\mathbb{R}$?

Now let us define a sum of two representations. First, we recall a construction from linear algebra.

Definition 2.5. *Suppose that V and U are vector spaces over $\mathbb{F}$. The Cartesian product $V \times U$ with respect to operations of component-wise addition and multiplication by scalars is called the (external) direct sum of V and U and is denoted by $V \oplus U$.*

Now we can define the sum of two representations of the same group over the same field.

Definition 2.6. *Assume that $T : G \to GL(V)$ and $S : G \to GL(U)$ are representations over $\mathbb{F}$. Then define $T \oplus S : G \to GL(V \oplus U)$ via*

$$(T \oplus S)(g)(v, u) := (T(g)v, S(g)u), \quad \forall g \in G, v \in V, u \in U. \tag{2.2}$$

We may suspect that a representation is a sum of some representations of smaller dimensions. How should we proceed to find those smaller blocks? It would be natural to give the following definition.

Definition 2.7. *Let $T : G \to GL(V)$ be a representation. A subspace $U \subset V$ is called invariant with respect to T if*

$$T(g)u \in U, \quad \forall g \in G, u \in U. \tag{2.3}$$

If $U \neq \{0\}$, $U \neq V$ then U is called non-trivial.

Note that we can construct the representation $S : G \to GL(U)$ setting $S(g)$ to be the restriction of the automorphism $T(g)$ to the subspace U. S is called a subrepresentation of the representation T.

Problem 2.8. *Describe the non-trivial invariant subspaces for the representations from Exercise 2.4.*

Problem 2.9. *Consider the representation $T : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2)$ given by the formula*

$$T(k) = \begin{pmatrix} \cos(\frac{2\pi k}{n}) & \sin\frac{2\pi k}{n}) \\ -\sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Find all its non-trivial invariant subspaces.

Question 2.10. *Assume $T : G \to GL(V)$ is a representation, $U \subset V$ is an invariant subspace, $u_1, \ldots, u_k$ is a basis of U, $u_1, \ldots, u_k, w_1, \ldots, w_m$ is a basis of V. We can express $T(g)$ in a block-diagonal form:*

$$T(g) = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}. \tag{2.4}$$

How can we reformulate the fact that U is an invariant subspace in terms of $A(g), B(g), C(g), D(g)$?

Definition 2.11. *Assume $T : G \to GL(V)$ is a representation. T is called irreducible if it has no non-trivial invariant subspaces.*

Definition 2.12. *Assume $T : G \to GL(V)$ is a representation. T is called completely reducible if for any invariant subspace U there exist another invariant subspace W such that $V = U \oplus W$. Here the notation $\oplus$ stands for another (internal) direct sum.*

Question 2.13. *Is it true that any irreducible representation is completely reducible?*

Question 2.14. *Does there exist a representation that is not completely reducible?*

Problem 2.15. *Define a representation T of the symmetric (permutation) group S_n in $\mathbb{C}^n$ by*

$$T(\sigma)((x_1, \ldots, x_n)) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

for $\sigma \in S_n$. Find all non-trivial invariant subspaces of the representation T.

3. Isomorphic representations

We are going to discuss when two representations $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ of the same group G over the same field $\mathbb{F}$ is natural to call isomorphic.

Question 3.1. *Let $T : G \to \mathrm{GL}(V)$ be a representation. Then V is also called G-module. Give a definition of a G-module so that every G-module corresponds to some representation and vice versa.*

Question 3.2. *Let G be a group. Give a definition of isomorphic G-modules.*

Question 3.3. *Suppose that $T : G \to \mathrm{GL}(V)$ and $S : G \to \mathrm{GL}(U)$ are representations over the same field $\mathbb{F}$. When would it be natural to say that T and S are isomorphic?*

4. Morphisms of representations and Schur's Lemma

Definition 4.1. *Assume that* $T : G \to \mathrm{GL}(V)$ *and* $S : G \to \mathrm{GL}(U)$ *are representations.* $A \in \mathrm{Hom}(V, U)$ *is called a morphism of representations* T *and* S *if*

$$AT(g) = S(g)A, \quad \forall g \in G. \tag{4.1}$$

We are going to denote the set of morphisms of representations T *and* S *by* $\mathrm{Mor}(T, S)$ *(by* $\mathrm{End}(T)$ *if* $S = T$*).*

Question 4.2. *How do we reformulate the definition of a morphism of representations if we consider those representations as* G*-modules?*

Question 4.3. *Is* $\mathrm{End}(T)$ *a subspace of* $\mathrm{End}(V)$ *?*

Problem 4.4. *Describe all morphisms between two one-dimensional complex representations of the group* $\mathbb{Z}_n$*.*

Definition 4.5. *Suppose that* $T : G \to \mathrm{GL}(V)$ *and* $S : G \to \mathrm{GL}(U)$ *are representations. They are called isomorphic if* $\exists A \in \mathrm{Iso}(V, U)$ *such that* $A \in \mathrm{Mor}(T, S)$ *(*A *is called an isomorphism of representations in that case). The set of isomorphisms of representations* T *and* S *is denoted by* $\mathrm{Iso}(T, S)$*. The set of automorphisms of the representation* T *is denoted by* $\mathrm{Aut}(T)$*.*

Lemma 4.6. *Suppose that* $T : G \to \mathrm{GL}(V)$ *and* $S : G \to \mathrm{GL}(U)$ *are representations,* $A \in \mathrm{Mor}(T, S)$*. Then*

a) $\ker A = \{v \in V \,|\, Av = 0\} \subset V$ *is an invariant subspace wrt* T*;*
b) $\mathrm{im} A = \{Av \,|\, v \in V\} \subset U$ *is an invariant subspace wrt* S*.*

Problem 4.7. *Prove this lemma.*

Proposition 4.8. *Suppose that* $T : G \to \mathrm{GL}(V)$ *and* $S : G \to \mathrm{GL}(U)$ *are irreducible representations,* $A \in \mathrm{Mor}(T, S)$*. Then either* $A = 0$ *or* $A \in \mathrm{Iso}(T, S)$*.*

Problem 4.9. *Prove this proposition.*

Proposition 4.10. *(Schur's Lemma) Suppose that* $T : G \to \mathrm{GL}(V)$ *is an irreducible complex representations, and* $A \in \mathrm{End}(T)$*. Then* $\exists \lambda \in \mathbb{C}$ *such that* $A = \lambda \mathrm{Id}$*.*

Problem 4.11. *Prove this proposition.*

Question 4.12. *Is the claim of Schur's Lemma still true over* $\mathbb{R}$ *?*

Definition 4.13. *A vector space* V *over a field* $\mathbb{F}$ *with an operation* $\cdot : V \times V \to V$ *is called an algebra if the following properties hold:*

(1) $a(b + c) = ab + ac, \quad \forall a, b, c \in V$;
(2) $(b + c)a = ba + bc, \quad \forall a, b, c \in V$;
(3) $\lambda(ab) = (\lambda a)b = a(\lambda b), \quad \forall a, b \in V, \forall \lambda \in \mathbb{F}$.

An algebra V *is called associative if* $a(bc) = (ab)c$, $\forall a, b, c \in V$*. An associative algebra* V *is called a division algebra if* $(V \setminus \{0\}, \cdot)$ *is a group.*

Question 4.14. *Assume* $T : G \to \mathrm{GL}(V)$ *is an irreducible representation over* $\mathbb{F}$*. Does* $\mathrm{End}(T)$ *have to be a division algebra over* $\mathbb{F}$ *?*

Chapters from the textbooks relevant to the lecture:

- Fulton, Harris, Section 1.1, 1.2;
- James, Liebeck, Chapters 1 - 5, 7 - 9;
- Vinberg, Sections 1.1 - 1.4.